Example Problems for Tricky Triangles

A home group activity

Directions

Students want to find as many unique triangles as possible, up to congruence. Let's start by showing them that the solution isn't always unique.

Example Exercise 1. A triangle with sides length $a = 5$ and $b = 7$ has an angle measuring $\alpha = \frac{\pi}{10}$ across from side a.

Solution. *Let β be the measure of the angle across from b. All triangles obey the law of sines, so we have:*

$$\frac{a}{\sin(\alpha)} = \frac{b}{\sin(\beta)} \Rightarrow \sin(\beta) = \frac{b}{a}\sin(\alpha) = \frac{7}{5}\sin\left(\frac{\pi}{10}\right) \doteq 0.4326.$$

Now, there are two co-terminal angles whose sine is 0.4326, *so we have*

$$\beta_1 \doteq 2.694 \doteq 154.4° \text{ or } \beta_2 \doteq 0.4474 \doteq 25.63°.$$

We can't exclude the possibility that β is an obtuse angle, since 7 could be the longest side of the triangle – it's longer than the other side we know, and we don't know about the other one yet.

Thus, the remaining angle $\gamma = \pi - \alpha - \beta$ also has two possible values:

$$\gamma_1 = \pi - \alpha - \beta_1 \doteq 0.1334 \doteq 7.645° \text{ or } \gamma_2 = \pi - \alpha - \beta_2 \doteq 2.380 \doteq 136.4°.$$

Invoking the law of sines again, the remaining side is:

$$\frac{c}{\sin(\gamma)} = \frac{a}{\sin(\alpha)} \Rightarrow c = a\frac{\sin(\gamma)}{\sin(\alpha)},$$

so it, too, two possible values:

$$c_1 = a\frac{\sin(\gamma_1)}{\sin(\alpha)} \doteq 2.153 \text{ or } c_2 = a\frac{\sin(\gamma_2)}{\sin(\alpha)} \doteq 11.16.$$

Therefore, there are two distinct triangles with the specified parameters. There aren't any others, since any other side lengths would violate the law of sines, and all triangles obey that.

Example Exercise 2. A triangle whose sides are 1, 3, and 3.

Solution. *This is a unique triangle. Any other triangle with those sides would be congruent to this one by SSS congruence theorem.*

We can even find the angles using the law of cosines. Let α be the measure of the angle across from the length-1 side. Then we have:

$$1^2 = 3^2 + 3^2 - 2 \times 3 \times 3 \cos(\alpha) = 18 - 18 \cos(\alpha) \Rightarrow \alpha = \arccos\left(\frac{17}{18}\right) \doteq 0.3349 \doteq 19.19^\circ$$

and, since this is an isosceles triangle:

$$\beta = \gamma \Rightarrow \beta = \frac{\pi - \alpha}{2} \doteq 1.403 \doteq 80.41^\circ \doteq \gamma.$$