Study Guide and Intervention

Roots and Zeros

Synthetic Types of Roots The following statements are equivalent for any polynomial function $f(x)$.

- c is a zero of the polynomial function $f(x)$.
- c is a root or solution of the polynomial equation $f(x) = 0$.
- $(x - c)$ is a factor of the polynomial $f(x)$.
- If c is real, then $(c, 0)$ is an intercept of the graph of $f(x)$.

Fundamental Theorem of Algebra	Every polynomial equation with degree greater than zero has at least one root in the set of complex numbers.
Corollary to the Fundamental Theorem of Algebras	A polynomial equation of the form $P(x) = 0$ of degree n with complex coefficients has exactly n roots in the set of complex numbers, including repeated roots.
Descartes' Rule of Signs	If $P(x)$ is a polynomial with real coefficients whose terms are arranged in descending powers of the variable, • the number of positive real zeros of $y = P(x)$ is the same as the number of changes in sign of the coefficients of the terms, or is less than this by an even number, and • the number of negative real zeros of $y = P(x)$ is the same as the number of changes in sign of the coefficients of the terms of $P(-x)$, or is less than this number by an even number.

Example 1 **Solve the equation $6x^3 + 3x = 0$. State the number and type of roots.**

$6x^3 + 3x = 0$

$3x(2x^2 + 1) = 0$

Use the Zero Product Property.

$3x = 0$ or $2x^2 + 1 = 0$

$x = 0$ or $2x^2 = -1$

$x = \pm \frac{i\sqrt{2}}{2}$

The equation has one real root, 0, and two imaginary roots, $\pm \frac{i\sqrt{2}}{2}$.

Example 2 **State the number of positive real zeros, negative real zeros, and imaginary zeros for $p(x) = 4x^4 - 3x^3 - x^2 + 2x - 5$.**

Since $p(x)$ has degree 4, it has 4 zeros.

Since there are three sign changes, there are 3 or 1 positive real zeros.

Find $p(-x)$ and count the number of changes in sign for its coefficients.

$p(-x) = 4(-x)^4 - 3(-x)^3 + (-x)^2 + 2(-x) - 5$

$= 4x^4 + 3x^3 + x^2 - 2x - 5$

Since there is one sign change, there is exactly 1 negative real zero.

Thus, there are 3 positive and 1 negative real zero or 1 positive and 1 negative real zeros and 2 imaginary zeros.

Exercises

Solve each equation. State the number and type of roots.

1. $x^2 + 4x - 21 = 0$

2. $2x^3 - 50x = 0$

3. $12x^3 + 100x = 0$

State the possible number of positive real zeros, negative real zeros, and imaginary zeros for each function.

4. $f(x) = 3x^3 + x^2 - 8x - 12$

5. $f(x) = 3x^5 - x^4 - x^3 + 6x^2 - 5$

Study Guide and Intervention *(continued)*

Roots and Zeros

Find Zeros

Complex Conjugate Theorem	Suppose a and b are real numbers with $b \neq 0$. If $a + bi$ is a zero of a polynomial function with real coefficients, then $a - bi$ is also a zero of the function.

Example **Find all of the zeros of $f(x) = x^4 - 15x^2 + 38x - 60$.**

Since $f(x)$ has degree 4, the function has 4 zeros.

$f(x) = x^4 - 15x^2 + 38x - 60$ $\quad f(-x) = x^4 - 15x^2 - 38x - 60$

Since there are 3 sign changes for the coefficients of $f(x)$, the function has 3 or 1 positive real zeros. Since there is + sign change for the coefficients of $f(-x)$, the function has 1 negative real zero. Use synthetic substitution to test some possible zeros.

2	1	0	−15	38	−60
		2	4	−22	32
	1	2	−11	16	−28

3	1	0	−15	38	−60
		3	9	−18	60
	1	3	−6	20	0

So 3 is a zero of the polynomial function. Now try synthetic substitution again to find a zero of the depressed polynomial.

−2	1	3	−6	20
		−2	−2	16
	1	1	−8	36

−4	1	3	−6	20
		−4	4	8
	1	−1	−2	28

−5	1	3	−6	20
		−5	10	−20
	1	−2	4	0

So − 5 is another zero. Use the Quadratic Formula on the depressed polynomial $x^2 - 2x + 4$ to find the other 1 zeros, $1 \pm i\sqrt{3}$.

The function has two real zeros at 3 and −5 and two imaginary zeros at $1 \pm i\sqrt{3}$.

Exercises

Find all zeros of each function.

1. $f(x) = x^3 + x^2 + 9x + 9$

2. $f(x) = x^3 - 3x^2 + 4x - 12$

3. $p(a) = a^3 - 10a^2 + 34a - 40$

4. $p(x) = x^3 - 5x^2 + 11x - 15$

5. $f(x) = x^3 + 6x + 20$

6. $f(x) = x^4 - 3x^3 + 21x^2 - 75x - 100$