1. (a) Try $\boxed{Y = 12X - 5}$ so that:

$$F(y) = P(Y \leq y) = P(12X - 5 \leq y) = P\left(X \leq \frac{y+5}{12}\right) = \int_{-\infty}^{\frac{y+5}{12}} dx f(x)$$

$$= \begin{cases} 0, & \frac{y+5}{12} < 0 \Leftrightarrow y < -5 \\ \frac{y+5}{12}, & 0 < \frac{y+5}{12} < 1 \Leftrightarrow -5 < y < 7 \\ 1, & \frac{y+5}{12} > 1 \Leftrightarrow y > 7 \end{cases}.$$

$$\therefore F'(y) = f(y) = \begin{cases} \frac{1}{12}, & -5 < y < 7 \\ 0, & \text{otherwise} \end{cases},$$

which is the density function of a uniform variable on $(-5, 7)$.

(b) Try $\boxed{Y = -2\ln X}$. Then:

$$F(y) = P(Y \leq y) = P(-2\ln X \leq y) = P\left(\ln X \geq \frac{-y}{2}\right) = P\left(X \geq e^{\frac{-y}{2}}\right)$$

$$= \int_{e^{\frac{-y}{2}}}^{\infty} dx f(x) = \begin{cases} 0, & e^{\frac{-y}{2}} > 1 \Leftrightarrow y < 0 \\ 1 - e^{\frac{-y}{2}}, & 0 < e^{\frac{-y}{2}} < 1 \Leftrightarrow \infty > y > 0 \\ 1, & e^{\frac{-y}{2}} < 0 \Leftrightarrow y > \infty \end{cases}.$$

$$\therefore F'(y) = f(y) = \begin{cases} \frac{e^{\frac{-y}{2}}}{2}, & y > 0 \\ 0, & y < 0 \end{cases},$$

which is the density function of an exponential variable with mean 2.

(c) Try $\boxed{Y = \sqrt[3]{X}}$. Then:

$$F(y) = P(Y \leq y) = P(\sqrt[3]{X} \leq y) = P\left(X \leq y^3\right) = \int_{-\infty}^{y^3} dx f(x)$$

$$= \begin{cases} 0, & y^3 < 0 \Leftrightarrow y < 0 \\ y^3, & 0 < y^3 < 1 \Leftrightarrow 0 < y < 1 \\ 1, & y^3 > 1 \Leftrightarrow y > 1 \end{cases}.$$

$$\therefore F'(y) = f(y) = \begin{cases} 3y^2, & 0 < y < 1 \\ 0, & \text{otherwise} \end{cases},$$

which is the density function sought.

2. We want to consider:

$$F(z) = P(Z \leq z) = P(X - Y \leq z) = P(Y \geq X - z),$$

which defines a half-plane h above the line $l = \{(X, Y) : Y = X - z\}$. Let $s = \{(X, Y) : (0 \leq X \leq 1) \wedge (0 \leq Y \leq 1)\}$ be the square on which $f(x, y)$ is supported. There are four cases to consider:

Case 1 ($z > 1$): The $X-$intercept of l is beyond $X = 1$, so that h contains the entirety of s so that:

$$F(z) = 1.$$

Case 2 ($0 < z < 1$): l cuts off an isosceles right triangle in the bottom-right corner of s a leg which measures $1 - z$. Thus:

$$F(z) = 1 - \frac{(1-z)^2}{2}.$$

Case 3 ($-1 < z < 0$): l cuts off all of s except an isosceles right triangle in the top-left corner a leg of which measures $1 + z$. Therefore:

$$F(z) = \frac{(1+z)^2}{2}.$$

Case 4 ($z < -1$): l is entirely above s so that $h \cap s = \emptyset$ and

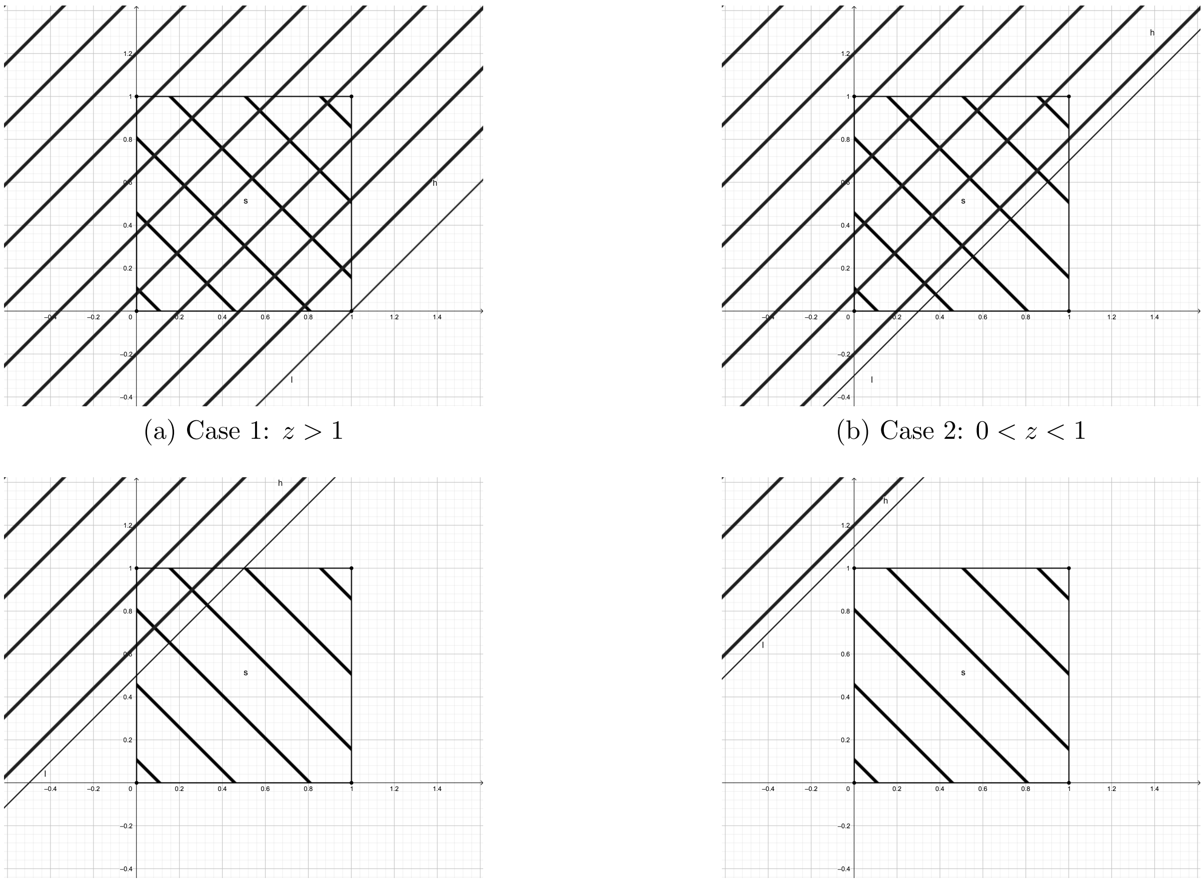
$$F(z) = 0.$$

In summary, then

$$F(z) = \begin{cases} 1, & z > 1 \\ 1 - \frac{(1-z)^2}{2}, & 0 < z < 1 \\ \frac{(1+z)^2}{2}, & -1 < z < 0 \\ 0, & z < -1 \end{cases},$$

so that

$$F'(z) = \boxed{f(z) = \begin{cases} 1 - z, & 0 < z < 1 \\ 1 + z, & 0 < z < -1 \\ 0, & \text{otherwise} \end{cases}}.$$



(a) Case 1: $z > 1$

(b) Case 2: $0 < z < 1$

(c) Case 3: $-1 < z < 0$

(d) Case 4: $z < -1$

Figure 1: The four cases for the position of l with respect to s. The area of the cross-hatched region is $F(z)$.

3. (a) The distribution function obeys:

$$F(z) = P(Z \leq z) = P(\sqrt{X^2 + Y^2} \leq z) = P(X^2 + Y^2 \leq z^2)$$

which, for $X > 0, Y > 0$ is the upper-right quadrant of a circle of radius z centered at the origin. Making the transformation $X = R\cos\Theta, Y = R\sin\Theta$ yields:

$$f(r, \theta) = 4r\cos\theta r\sin\theta e^{-r^2}$$

so that the distribution function of Z is:

$$F(z) = \int_0^{\frac{\pi}{2}} d\theta \int_0^z drr \left(4r\cos\theta r\sin\theta e^{-r^2}\right) = \int_0^{\frac{\pi}{2}} d\theta 2\cos\theta\sin\theta \int_0^z dr 2r^3 e^{-r^2}$$

$$= \int_0^{\frac{\pi}{2}} d\theta sin(2\theta) \left(\left[-r^2 e^{-r^2}\right]_0^z + \int_0^z dr 2re^{-r^2} \right)$$

$$= \int_0^{\pi} \frac{d\phi}{2} sin\phi \left(-z^2 e^{-z^2} + \left[-e^{-r^2}\right]_0^z\right) = \left[\frac{-\cos\phi}{2}\right]_0^{\pi} \left(z^2 e^{-z^2} - e^{-z^2} + 1\right)$$

$$= \boxed{1 - (z^2+1)e^{-z^2}}.$$

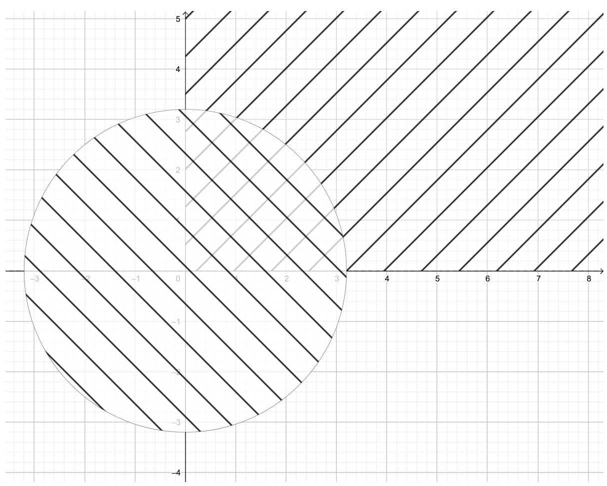


Figure 2: The region of integration is cross-hatched.

(b) As we know the distribution function from part (a), we can find the probability density of Z as:

$$f(z) = F'(z) = 0 - (2z+0)e^{-z^2} - (z^2+1)e^{-z^2}(-2z)$$
$$= -2ze^{-z^2} + 2z^3e^{-z^2} + 2ze^{-z^2}$$
$$= \boxed{2z^3e^{-z^2}}.$$

4. The distribution function for Z is:

$$F(z) = P(Z \leq z) = P\left(\frac{X+Y}{2} \leq z\right) = P(Y \leq -X + 2z),$$

so that the area of integration is the half-plane below the line $l = \{(X,Y) : Y = -X + 2z\}$.

Since $f(x,y)$ is supported only in the first quadrant, we have:

$$F(z) = 0, z < 0$$

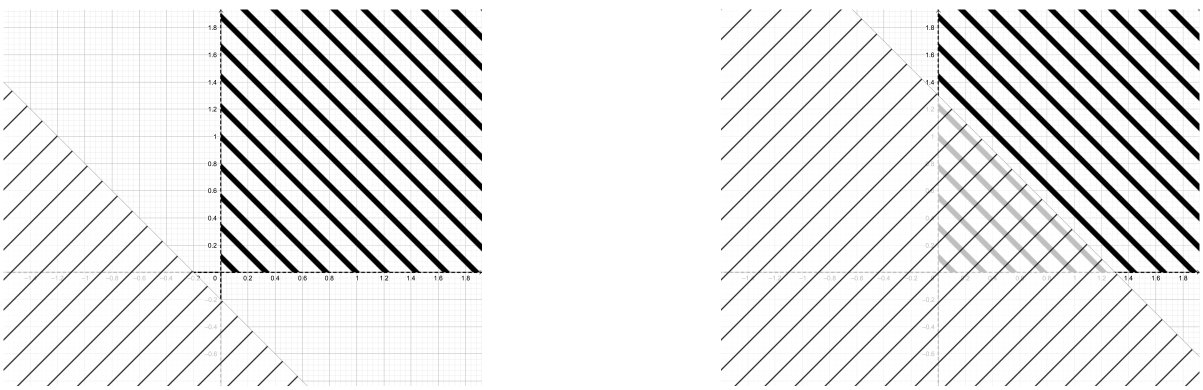
as in that case, l is entirely in quadrants II-IV.

On the other hand, if $z > 0$, we have:

$$F(z) = \int_0^{2z} dx \int_0^{2z-x} dy e^{-x-y} = \int_0^{2z} dx e^{-x} \int_0^{2z-x} dy e^{-y} = \int_0^{2z} dx e^{-x} \left[\frac{e^{-y}}{-1}\right]_0^{2z-x}$$

$$= \int_0^{2z} dx e^{-x}\left(1 - e^{-2z+x}\right) = \int_0^{2z} dx e^{-x} - \int_0^{2z} dx e^{-2z} = \left[\frac{e^{-x}}{-1}\right]_0^{2z} - 2ze^{-2z}$$

$$= 1 - e^{-2z} - 2ze^{-2z} = 1 - (2z+1)e^{-2z}.$$



(a) $z < 0$ (b) $z > 0$

Figure 3: The two cases for the position of l. The cross-hatched region is the area of integration for $F(z)$.

Therefore, the distribution function of Z is:

$$F(z) = \begin{cases} 1 - (2z+1)e^{-2z}, & z > 0 \\ 0, & z < 0 \end{cases}$$

Therefore, we can find the probability density of Z as:

$$f(z) = F'(z) = \begin{cases} 0 - (2)e^{-2z} - (2z+1)e^{-2z}(-2), & z > 0 \\ 0, z < 0 \end{cases}$$

$$= \begin{cases} (-2 + 4z + 2)e^{-2z}, & z > 0 \\ 0, & z < 0 \end{cases}$$

$$= \boxed{\begin{cases} 4ze^{-2z}, & z > 0 \\ 0, & z < 0 \end{cases}}.$$