1. Let X, Y, and Z be the lifetimes of each battery in proper units. Let $W = \max(X, Y, Z)$ so that:

$$F(w) = P(W \leq w) = P[\max(X, Y, Z) \leq w] = P(X \leq w, Y \leq w, Z \leq w).$$

Assume X, Y, and Z are independent (this is only explicitly stated in part 3, but is needed for all parts) so that:

$$F(w) = P(X \leq w)P(Y \leq w)P(Z \leq w).$$

Further, since X, Y, and Z are identical in all cases:

$$F(w) = [P(X \leq w)]^3.$$

a)

$$P(X \leq w) \overset{0<w<1}{=} \int_0^w dx = w \Rightarrow F(w) = P^3(X \leq w) = \begin{cases} 0, & w < 0 \\ w^3, & 0 < w < 1 \\ 1, & 1 < w \end{cases}.$$

$$\therefore f(w) = F'(w) = \begin{cases} 3w^2, & 0 < w < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\therefore E(W) = \int_{-\infty}^{\infty} wf(w)dw = \int_0^1 3w^3 dw = \frac{3}{4}w^4\Big|_0^1 = \frac{3}{4}.$$

Thus the machine's expected lifetime is $\boxed{\frac{3}{4}\text{ yr.}}$

b)

$$P(X \leq w) \overset{0 \leq w}{=} \int_0^w e^{-x}dx = \frac{e^{-x}}{-1}\Big|_0^w = 1 - e^{-w} \Rightarrow F(w) = \begin{cases} 0, & w < 0 \\ (1 - e^{-w})^3, & 0 < w \end{cases}.$$

$$\therefore f(w) = F'(w) = \begin{cases} 3(1 - e^{-w})^2 e^{-w}, & 0 < w \\ 0, & \text{otherwise} \end{cases}.$$

$$\therefore E(W) = \int_{-\infty}^{\infty} wf(w)dw = \int_0^{\infty} 3we^{-w}(1 - e^{-w})^2 dw = 3\int_0^{\infty} \left(w - 2we^{-w} + we^{-2w}\right) dw$$

$$= 3\left[-(1+z)e^{-z} + \frac{1+2z}{2}e^{-2z} - \frac{1+3z}{9}e^{-3z}\right]_0^{\infty} = 3\left(1 - \frac{1}{2} + \frac{1}{9}\right) = \frac{11}{6}.$$

Thus the machine's expected lifetime is $\boxed{1\frac{5}{6}\text{ yr.}}$

c)

$$P(X \le w) \overset{0<w<1}{=} \int_0^w (2x)dx = \left.\frac{2x^2}{2}\right|_0^w = w^2 \Rightarrow F(w) = \begin{cases} 0, & w<0 \\ w^6, & 0<w<1 \\ 1, & 1<w \end{cases}.$$

$$\therefore f(w) = F'(w) = \begin{cases} 6w^5, & 0<w<1 \\ 0, & \text{otherwise} \end{cases}.$$

$$\therefore E(W) = \int_{-\infty}^{\infty} wf(w)dw = \int_0^1 6w^6 dw = \left.\frac{6}{7}w^7\right|_0^1 = \frac{6}{7}.$$

Thus the machine's expected lifetime is $\boxed{\frac{6}{7}}$.

2. Let X and Y be the lifetimes of the parts in the appropriate unit. Then the machine's lifetime in the same units is

$$Z = \min(X, Y).$$

This has distribution function:

$$F(z) = P(Z \le z) = P[\min(X,Y) \le z] = P(X \le z) + P(Y \le z) - P(X \le z, Y \le z).$$

a) In this case, the part lifetimes are identical and independent, so that:

$$P(Y \le z) = P(X \le z) \text{ and } P(X \le z, Y \le z) = [P(X \le z)]^2$$

and we need compute only $P(X \le z)$.

$$P(X \le z) \overset{0<z<2}{=} \int_0^z \frac{1}{2}dx = \frac{z}{2} \Rightarrow F(z) = 2P(X \le z) - P^2(X \le z) = \begin{cases} 0, & z<0 \\ z - \frac{z^2}{4}, & 0<z<2 \\ 1, & 2<z \end{cases}.$$

$$\therefore f(z) = F'(z) = \begin{cases} \frac{2-z}{2}, & 0<z<2 \\ 0, & \text{otherwise} \end{cases}.$$

$$\therefore E(Z) = \int_{-\infty}^{\infty} zf(z)dz = \int_0^2 \left(z - \frac{z^2}{2}\right) dz = \left[\frac{z^2}{2} - \frac{z^3}{6}\right]_0^2 = \frac{2}{3}.$$

The machine's expected lifetime is therefore $\boxed{\frac{2}{3}\text{ yr.}}$

b) Once again, X and Y are identical and independent, so that:

$$P(Y \le z) = P(X \le z) \text{ and } P(X \le z, Y \le z) = [P(X \le z)]^2$$

and we need compute only $P(X \le z)$.

$$P(X \le z) \overset{0 \le z}{=} \int_0^z \frac{e^{-x/2}}{2} dx = \left. \frac{e^{-x/2}/2}{-1/2} \right|_0^z = 1 - e^{-z/2}$$

$$\therefore F(z) = 2P(X \le z) - P^2(X \le z) = \begin{cases} 2(1 - e^{-z/2}) - (1 - e^{-z/2})^2, & 0 < z \\ 0, & z < 0 \end{cases}$$

$$= \begin{cases} 1 - e^{-z}, & z > 0 \\ 0, & \text{otherwise} \end{cases}.$$

$$\therefore f(z) = F'(z) = \begin{cases} e^{-z}, & 0 < z \\ 0, & \text{otherwise} \end{cases}.$$

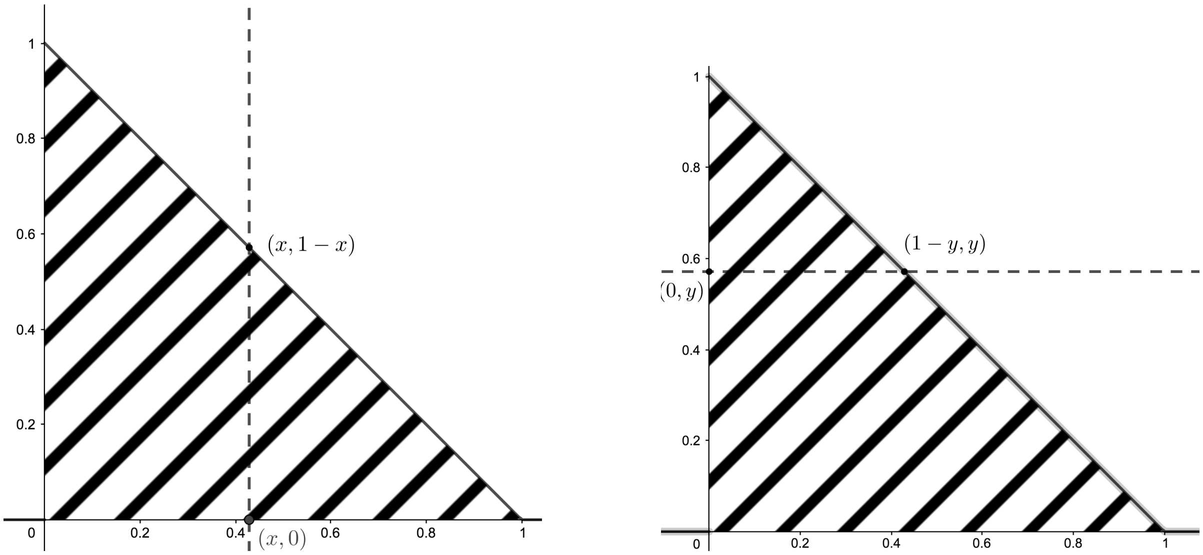
$$\therefore E(Z) = \int_{-\infty}^{\infty} z f(z) dz = \int_0^{\infty} z e^{-z} dz = \left. \frac{z e^{-z}}{-1} \right|_0^{\infty} + \int_0^{\infty} e^{-z} dz = 1$$

The machine's expected lifetime is therefore $\boxed{1 \text{ yr.}}$

c) In this case, it is not obvious that X and Y are identical and it seems likely they are not independent. Let's start by computing the marginal densities:

$$f(x) = \int_{-\infty}^{\infty} f(x,y) dy \overset{0<x<1}{=} \int_0^{1-x} 2 dy = 2(1-x) \Rightarrow f(x) = \begin{cases} 2 - 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}.$$

$$g(y) = \int_{-\infty}^{\infty} f(x,y) dx \overset{0<y<1}{=} \int_0^{1-y} 2 dx = 2 - 2y \Rightarrow g(y) = \begin{cases} 2 - 2y, & 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}.$$



(a) $f(x)$ (b) $g(y)$

Figure 1: Lines of integration used to determine marginal densities. The shaded triangle is the support of $f(x, y)$.

As $f(x) = g(x)$ but $f(x)g(y) \neq f(x, y)$, X and Y are identical but not independent. Therefore, $P(X \leq z) = P(Y \leq z) \neq \sqrt{P(X \leq z, Y \leq z)}$ and we need to compute $P(X \leq z)$ and $P(X \leq z, Y \leq z)$:

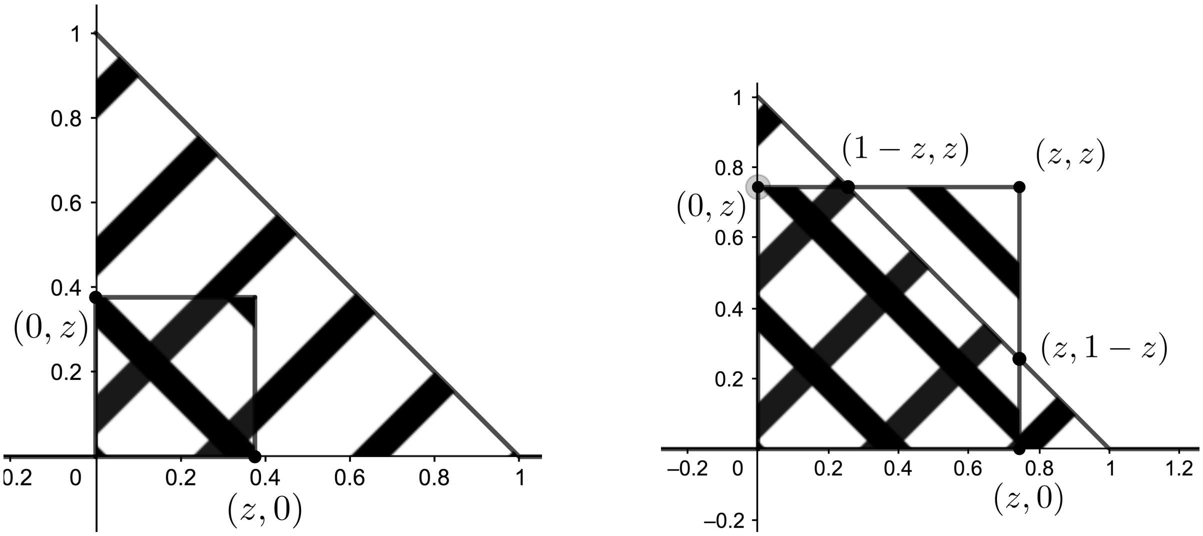
$$P(X \leq z) \overset{0<z<1}{=} \int_0^z (2 - 2x)dx = 2x - x^2\Big|_0^z = 2z - z^2$$

$$\therefore P(X \leq z) = \begin{cases} 0, & z < 0 \\ 2z - z^2, & 0 < z < 1 \\ 1, & 1 < z \end{cases}$$

$$P(X \leq z, Y \leq z) \overset{0<z<1/2}{=} 2z^2.$$

$$P(X \leq z, Y \leq z) \overset{1/2<z<1}{=} 2\left[z^2 - \frac{(2z-1)^2}{2}\right] = 4z - 2z^2 - 1.$$

$$\therefore P(X \leq z, Y \leq z) = \begin{cases} 0, & z < 0 \\ 2z^2, & 0 < z < \frac{1}{2} \\ 4z - 2z^2 - 1, & \frac{1}{2} < z < 1 \\ 1, & 1 < z \end{cases}.$$



(a) $0 < z < \frac{1}{2}$ (b) $\frac{1}{2} < z < 1$

Figure 2: $P(X \leq z, Y \leq z)$ is twice the cross-hatched area.

Thus we have at last:

$$\therefore F(z) = 2P(X \leq z) - P(X \leq z, Y \leq z) = \begin{cases} 0, & 0 < z \\ 4z - 2z^2 - 2z^2, & 0 < z < \frac{1}{2} \\ 4z - 2z^2 - 4z + 2z^2 + 1, & \frac{1}{2} < z < 1 \\ 1, & 1 < z \end{cases}$$

$$= \begin{cases} 0, & z < 0 \\ 4z - 4z^2, & 0 < z < \frac{1}{2} \\ 1, & \frac{1}{2} < z \end{cases}$$

$$\therefore f(z) = F'(z) = 4 \begin{cases} 1 - 4z, & 0 < z < \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}.$$

$$\therefore E(Z) = \int_{-\infty}^{\infty} z f(z) dz = \int_{0}^{1/2} z(4 - 8z) dz = \left[4\frac{z^2}{2} - 8\frac{z^3}{3} \right]_0^{1/2} = \frac{1}{6}.$$

Therefore, the machine's mean lifetime is $\boxed{\frac{1}{6}}$.

3. For $W = \min(X, Y, Z)$ we have:

$$\begin{aligned} F(w) =& P(W \leq w) = P[\min(X, Y, Z) \leq w] = P[(X \leq w) \vee (Y \leq w) \vee (Z \leq w)] \\ =& P(X \leq w) + P(Y \leq w) + P(Z \leq w) \\ & - P(X \leq w, Y \leq w) - P(X \leq w, Z \leq w) - P(Y \leq w, Z \leq w) \\ & + P(X \leq w, Y \leq w, Z \leq w). \end{aligned}$$

As in all cases X, Y, and Z are identical and independent, this reduces to:

$$F(w) = 3P(X \leq w) - 3[P(X \leq w)]^2 + [P(X \leq w)]^3$$

and we need compute only $P(X \leq w)$.

a)

$$P(X \leq w) \overset{0<w<1}{=} \int_0^w 1dx = w \Rightarrow F(w) = \begin{cases} 3w - 3w^2 + w^3, & 0 < w < 1 \\ 0, & \text{otherwise} \end{cases}.$$

$$\therefore F'(w) = \boxed{f(w) = \begin{cases} 3 - 6w + 3w^2, & 0 < w < 1 \\ 0, & \text{otherwise} \end{cases}}.$$

b)

$$P(X \leq w) \overset{0\leq w}{=} \int_0^w e^{-x}dx = \left.\frac{e^{-x}}{-1}\right|_0^w = 1 - e^{-w}$$

$$\begin{aligned} \therefore F(w) &\overset{0\leq w}{=} 3(1 - e^{-w}) - 3(1 - e^{-w})^2 + (1 - e^{-w})^3 \\ &\overset{0\leq w}{=} 3 - 3e^{-w} - 3 + 6e^{-w} - 3e^{-2w} + 1 - 3e^{-w} + 3e^{-2w} - e^{-3w} \\ &\overset{0\leq w}{=} 1 - e^{-3w} \Rightarrow F(w) = \begin{cases} 1 - e^{-3w}, & 0 < w \\ 0, & \text{otherwise} \end{cases}. \end{aligned}$$

$$\therefore F'(w) = \boxed{f(w) = \begin{cases} 3e^{-3w}, & 0 < w \\ 0, & \text{otherwise} \end{cases}}.$$

4. Let X_i be the i^{th} dice roll so that the maximum of two rolls is $S = \max(X_1, X_2)$. Then:

$$F(s) = P(S \leq s) = P[\max(X_1, X_2) \leq s] = P(X_1 \leq s, X_2 \leq s).$$

As X_1 and X_2 are independent and identical:

$$P(X_1 \leq s, X_2 \leq s) = P(X_1 \leq s)P(X_2 \leq s) = [P(X_1 \leq s)]^2.$$

$$P(X_1 \leq s) = \sum_{x=0}^{s} g(x) = \begin{cases} 0, & s = 0 \\ \frac{1}{6}, & s = 1 \\ \vdots & \\ 1, & s \geq 6 \end{cases} = \begin{cases} 0, & s = 0 \\ \frac{s}{6}, & 1 \leq s \leq 5 \\ 1, & s \geq 6 \end{cases}.$$

$$\therefore F(s) = \begin{cases} 0, & s = 0 \\ \frac{s^2}{36}, & 1 \leq s \leq 5 \\ 1, & s \geq 6 \end{cases}$$

We can compute the mass function for S by forming sequential differences in $F(s)$ such that $f(s) = F(s) - F(s-1)$ and $f(0) = 0$:

$$f(s) = \begin{cases} 0, & s = 0 \\ 1/36 - 0, & s = 1 \\ 4/36 - 1/36, & s = 2 \\ 9/36 - 4/36, & s = 3 \\ \vdots & \\ 36/36 - 25/36, & s = 6 \\ 0, & s > 6 \end{cases} = \begin{cases} \frac{2s-1}{36}, & s = 1, 2, ..., 6 \\ 0, & \text{otherwise} \end{cases}.$$

$$\therefore E(S) = \sum_{s=0}^{\infty} s f(s) = 0 \cdot 0 + 1 \cdot \frac{1}{36} + 2 \cdot \frac{3}{36} + ... + 6 \cdot \frac{11}{36} = \frac{161}{36} = \boxed{4\frac{17}{36}}.$$