Mike Sorice — College Geometry II
Spring 2019 — Test 1

1.

Proposition. *Given* $\square ABCD$, $[(AC \cap BD = \{E\}) \wedge (BE \cong CE) \wedge (AE \cong DE)] \Rightarrow (\angle BAD \cong \angle CDA)$.

Proof. $\angle CED \cong \angle AEB$ as vertical angles.

$\triangle AEB \cong \triangle CED$ by S.A.S. ($AE \cong DE$ by hypothesis, $\angle AEB \cong \angle DEC$ by previous step, and $BE \cong CE$ by hypothesis.)

$\therefore AB \cong CD$ as corresponding parts of congruent triangles.

$[(AE \cong DE) \wedge (BE \cong CE) \wedge (A * E * C) \wedge (B * E * D)] \Rightarrow (AC \cong BD)$ by segment addition.

$\therefore \triangle ABD \cong \triangle DCA$ by S.S.S. (AD is a common side, $AB \cong CD$ by previous step, $AC \cong BD$ by previous step.)

$\therefore \angle BAD \cong \angle ADC$ as corresponding parts of congruent triangles. $\square$

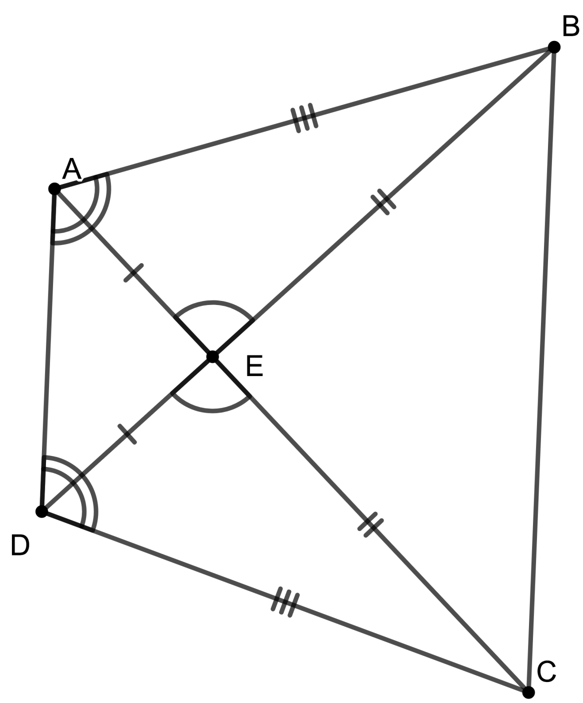


Figure 1: Quadrilateral whose diagonals bisect each other.

2.

Proposition. *Given* $\square ABCD$, $[(AC \cap BD = \{E\}) \wedge (AC \cong BD) \wedge (AB \cong CD)] \Rightarrow [(AE \cong DE) \wedge (BE \cong CE)]$.

Proof. $\triangle ADC \cong \triangle DAB$ by S.S.S. (AD is a common side, $AB \cong DC$ by hypothesis, and $AC \cong BD$ by hypothesis.)

$\therefore \angle CAD \cong \angle BDA$ as corresponding parts of congruent triangles.

$\therefore \triangle ADE$ is isosceles, having congruent base angles, so it also has congruent sides opposite, i.e. $AE \cong DE$.

Similarly, $\triangle ABC \cong \triangle DCB$ by S.S.S. (BC is a common side, $AB \cong CD$ by hypothesis, and $AC \cong BD$ by hypothesis.)

$\therefore \angle ACB \cong \angle CBD$ as corresponding parts of congruent triangles.

$\therefore \triangle BCE$ is isosceles, having congruent base angles, so it also has congruent sides opposite by theorem, i.e. $BE \cong CE$. □

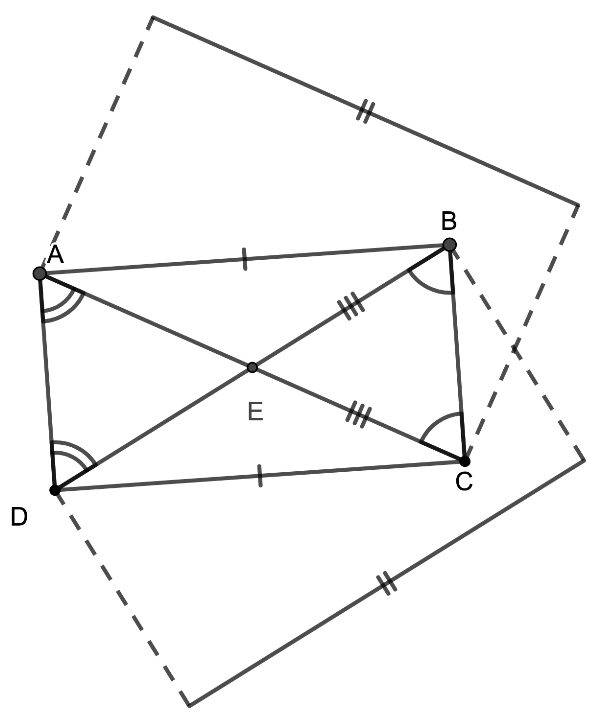


Figure 2: Quadrilateral whose diagonals and a pair of opposite sides are congruent.

3.

Proposition. *Given* $\square ABCD$, $[(AC \cap BD = \{E\}) \wedge (AB \cong BC) \wedge (AD \cong CD)] \Rightarrow [(AE \cong CE) \wedge (BD \perp AC)]$.

Proof. $\triangle ABD \cong \triangle CBD$ by S.S.S. ($AB \cong BC$ by hypothesis, BD is a common side, and $AD \cong CD$ by hypothesis.)

$\therefore \angle ABD = \angle ABE \cong \angle CBD = \angle CBE$ as corresponding parts of congruent angles and as $B * E * D$.

$\therefore \triangle ABE \cong \triangle CBE$ by S.A.S. ($AB \cong BC$ by hypothesis, $\angle ABE \cong \angle CBE$ by previous step, and BE is a common side.)

$\therefore AE \cong CE$ as corresponding parts of congruent triangles. //

Likewise, $\angle AEC \cong \angle BEC$.

$\angle AEC$ supplements $\angle BEC$ by definition.

$\therefore \angle AEC$ and $\angle BEC$ are right angles by definition (congruent to supplement.)

$\therefore AC \perp BD$ by definition (segments form a right angle.) $\square$

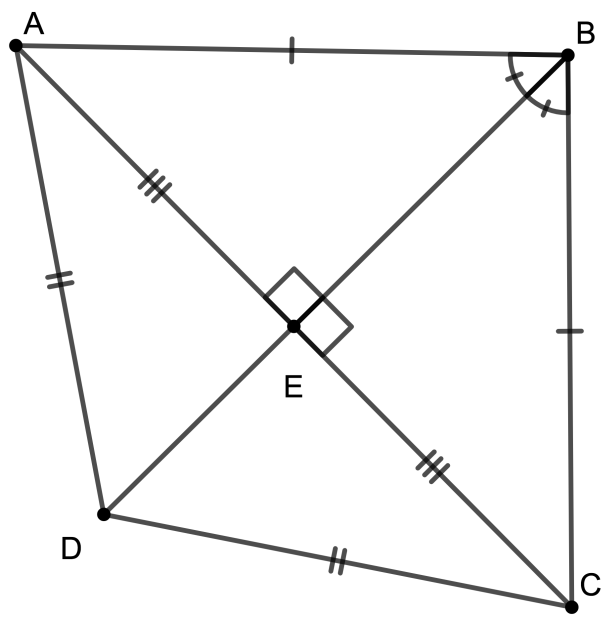


Figure 3: Quadrilateral with two pairs of congruent adjacent sides.

4.

Proposition. *Given* $\square ABCD$, $[(AC \perp BD) \wedge (AB \cong BC)] \Rightarrow (AD \cong CD)$.

Proof. $\angle AEB, \angle BEC, \angle CED$, and $\angle AED$ are all right angles by definition of perpendicular segments.

$\therefore \angle AEB \cong \angle BEC \cong \angle CED \cong \angle AED$ as all right angles are congruent.

$\triangle ABE \cong \triangle CBE$ by H.L.[1] ($\angle AEB$ and $\angle BEC$ are both right angles by previous step, so AB and BC are hypotenuses of right triangles $\triangle ABE$ and $\triangle CBE$, respectively; $AB \cong BC$ by hypothesis; and BE is a common leg.)

[1]H.L. congruence is Greenberg's Proposition 4.6. We'll also prove it here in Problem 6.

$\therefore AE \cong CE$ as corresponding parts of congruent triangles.

$\therefore \triangle ADE \cong \triangle CDE$ by S.A.S. ($AE \cong CE$ by previous step, $\angle AED \cong \angle CED$ by previous step, and DE is a common side.)

$\therefore AD \cong CD$ as corresponding parts of congruent triangles. $\square$

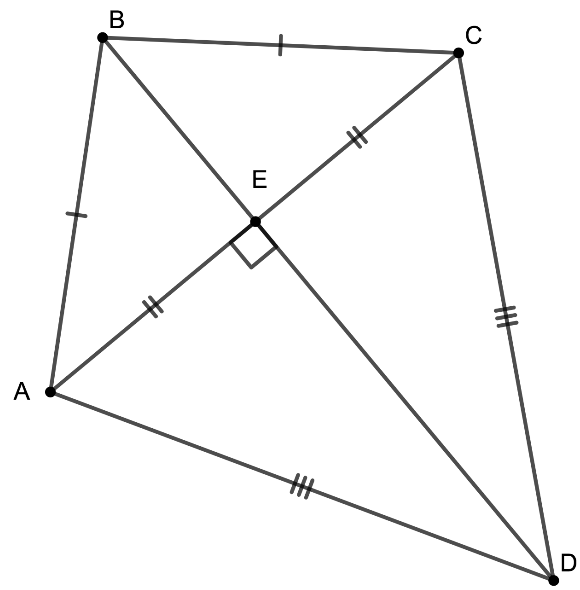


Figure 4: Quadrilateral with perpendicular diagonals and pair of congruent adjacent sides.

5. Given $\triangle ABC$ with $\overline{AB} = c > \overline{AC} = b > \overline{BC} = a$, $\angle C^{\circ} = \gamma > e$, $\{X\} = \circ(A, AC) \cap AB$,$\{Y\} = \circ(B, BC) \cap AB$, $\overline{CX} = x$, $\overline{CY} = y$, and $\overline{XY} = z$...

   (a)

   Proposition. $\triangle AXY$ *is acute.*

   Proof. By construction, $AX \cong AC$ – both are radii of $\circ(A, AC)$.

   $\therefore \triangle ACX$ is isosceles with legs AC and AX.

   $\therefore \angle ACX \cong \angle AXC$

   Similarly, $BC \cong BY \Rightarrow \angle BCY \cong \angle BYC$.

   Consider $\triangle ACX$. Its exterior angle at X is $\angle BCX$, which supplements $\angle AXC$. Therefore, by the exterior angle theorem, remote interior $\angle ACX < \angle BXC$.

   But $\angle ACX \cong \angle AXC$ by previous step, so $\angle AXC$ is less than its complement.

   $\therefore \angle AXC$ is acute by definition.

   A similar application of the E.A.T. considering $\angle AYC$ exterior to $\triangle BCY$ at Y gives $\angle BYC < \angle AYC \Rightarrow \angle BYC$ is acute.

$\angle XCY$ is within $\angle ACX$, so $\angle XCY < \angle ACX$ by definition. $\therefore$ $\angle XCY$ is also acute (an angle less than an acute angle is also acute.)[2]

All angles of $\triangle CXY$ being therefore acute, $\triangle CXY$ is an acute triangle by definition. $\square$

(b)

Proposition. *$z < x$.*

Proof. Since $\angle XCY$ is within $\angle ACX$, $\angle XCY < \angle ACX$ by definition. However, $\angle ACX \cong \angle AXC$ by previous step, so $\angle XCY < \angle ACX$.

The longer side of a triangle being opposite the larger angle, $XY < CY \Rightarrow \overline{XY} = z < y = \overline{CY}$ (Greenberg's Proposition 4.5.) $\square$

Proposition. *$z < y$.*

Proof. Similarly, $\angle XCY < \angle BYC \Rightarrow z < x$. $\square$

Proposition. *$z = a + b - c$.*

Proof. By construction, $c = a + b - z$.

Subtracting c from and adding z to both sides yields: $c - c + z = z = a + b - c = a + b - z + z - c$. $\square$

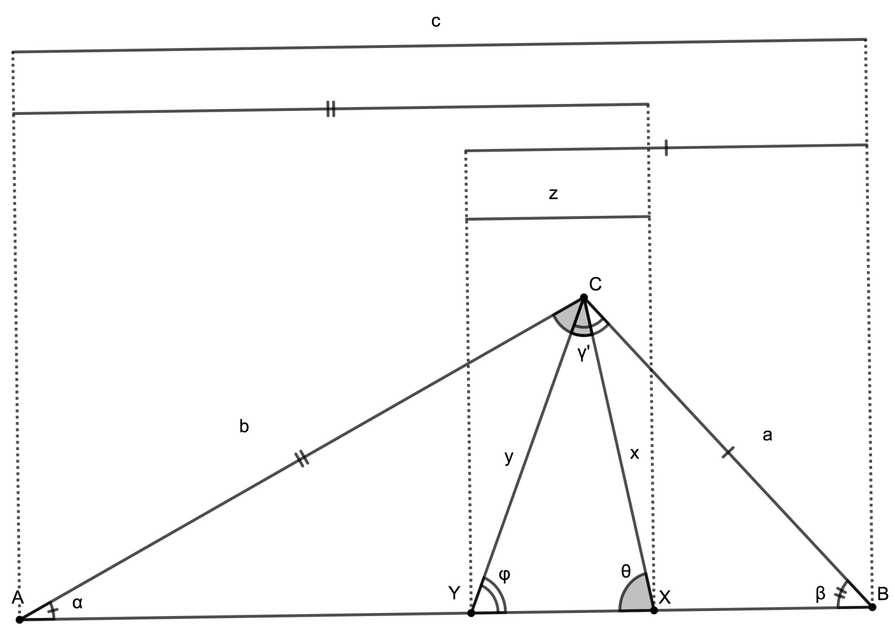


Figure 5: Obtuse scalene triangle with smaller inner triangle formed by laying off shorter legs on longest.

[2]It would work equally well to note that $\angle XCY$ is also within and therefore smaller than acute $\angle BCY$.

(c)

Proposition. *$y > x$.*

Proof by G. Galperin. (-3) By definition, $\triangle ACX$ is isosceles with leg b and $\triangle BCX$ is isosceles with leg a.
Thus, $\angle BCY \cong \angle BYC$ and both are acute (as otherwise $\sum(\triangle BCY) > 2e$, contradicting the Saccheri-Legendre theorem.)
Similarly, $\angle ACX \cong \angle AXC$ and both are acute.

(-2) Further, since $a + b > c = \overline{AB}$ by the triangle inequality on $\triangle ABC, A * Y * X * B$ (as exactly one of $A * X * Y * B, X = Y$, or $A * Y * X * B$ but if $X = Y, a + b = c$ and if $A * X * Y * B, a + b < c$.)

(-1) $AC > BC \Rightarrow \angle B > \angle A \Rightarrow \beta > \alpha$ as angles opposite larger sides are larger.

(0) Drop a perpendicular from C to AB with foot H. Then exactly one of $H * Y * XH = Y, Y * H * X, H = X$, or $Y * X * H$.

Case 1 ($H*Y*X$): Then $\angle CYX$ is exterior to $\triangle CHY$ and thus $\angle CYX > \angle CHY$, but $\angle CYX$ is acute by (-3). $\Rightarrow\!\Leftarrow$

Case 2 ($Y * X * H$): Similarly, acute $\angle CXY$ is remote to, and therefore bigger than, right $\angle CHX$ in $\triangle CHX$. $\Rightarrow\!\Leftarrow$

Case 3 ($H = Y$): Then $\angle CYX \cong \angle CHX$, but the former is acute by (-3), while the latter is right by construction. $\Rightarrow\!\Leftarrow$

Case 4 ($H = X$): Similarly, acute $\angle CXY \cong \angle CHY$, a right angle. $\Rightarrow\!\Leftarrow$

$\therefore Y * H * X$.

(1) Lay off a copy of AH on $-\overrightarrow{HA}$, such that $D \in -\overrightarrow{HA}$ and $DH \cong AH$. Then $\triangle ACH \cong \triangle DCH$ by side-angle-side congruence (CH is a shared side, $\angle AHC \cong \angle DHC$ as both are right angles by construction at (0), $AH \cong DH$ by construction.)
$\therefore \overline{CD} = b$ and $\angle HDC^\circ = \alpha$ by congruent triangles.

(2) Exactly one of $H * D * B$, $D = B$, or $H * B * D$ by betweenness axioms.

Case 1 ($D = B$): $CD \cong BC \Rightarrow a = b$, contradicting the hypothesis $b > a$. $\Rightarrow\!\Leftarrow$

Case 2 ($H * D * B$): Then $\angle HDC$ is an exterior angle of $\triangle BCD$, so $\angle CDH > \angle HBC$ by the exterior angle theorem, i.e. $\alpha > \beta$, contradicting (-1). $\Rightarrow\!\Leftarrow$

$\therefore H * B * D$.

(3) By the exterior angle theorem on $\triangle BCH, \angle CBD > \angle CHB$ and, as the latter is a right angle, $\angle CBD$ is obtuse, i.e. $\angle CBD^\circ = \gamma > e$.

(4) Lay off HX on $\overrightarrow{HA}$ to create $Z \in \overrightarrow{HA}$ such that $HZ \cong HX$. By betweenness axioms, exactly one of $Z * Y * H, Y = Z$, or $Y * Z * H$.

(5) $\triangle CHZ \cong \triangle CHX$ by side-angle-side congruence (CH is a common side, $\angle CHZ \cong \angle CHX$ as both are right angles by construction at (0), $HX \cong HZ$ by construction at (4).) Therefore $CZ \cong CX$ by congruent triangles.

(6) $\overline{AX} = b$ by hypothesis.
$\overline{AX} = \overline{AH} + \overline{HX}$ since $A * Y * H * X * B$ by (0) and (-2).
$DH \cong AH$ by construction at (1).
$HZ \cong HX$ by construction at (4).
$\therefore \overline{AX} = \overline{AH} + \overline{HX} = \overline{DH} + \overline{HZ}$, but since $Z * H * D$ by construction at (4), $\overline{DH} + \overline{HZ} = \overline{DZ}$.
$\therefore DZ \cong AX$ and $\overline{DZ} = \overline{AX} = b$.

(7) $\overline{BD} + \overline{BC} > \overline{CD}$ by the triangle inequality on $\triangle BCD$.
$BC \cong BY$ by hypothesis, so $\overline{BD} + \overline{BY} > \overline{CD}$.
$\overline{AC} = \overline{CD} = \overline{DZ}$ by (1) and (5).
$\therefore \overline{BD} + \overline{BY} > \overline{DZ}$

(8) $Y * B * D$ by (1), so $\overline{BD} + \overline{BY} = \overline{DY}$.
$\therefore DY > DZ$ by (7).
$\therefore Y * Z * D$. As $Z \in \overrightarrow{HA}$ by construction at (4) and $Y * H * Z$ by (0) and (1), $Y * Z * H$.

(9) $\therefore HY > HZ$.
$HZ \cong HX$ by construction at (4), so $HY > HS$.

(10) $\therefore CY > CX$ by the same reasoning as (2).

□

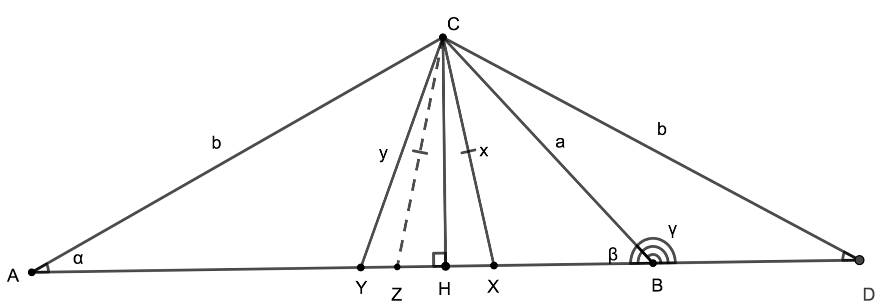


Figure 6: Construct $\triangle DCH \cong \triangle ACH$ to show that $\triangle CHZ \cong \triangle CHY$, so $y > x$.

6.

Proposition. *Given* $\triangle ABC$ *and* $\triangle A'B'C'$, $\{[\angle C^\circ = (\angle C')^\circ = e] \wedge [\overline{BC} = \overline{B'C'} = a] \wedge [\overline{AB} = \overline{A'B'} = c]\} \Rightarrow (\triangle ABC \cong \triangle A'B'C')$.

Proof by construction: Lay off $B''C'' \cong B'C'$ on $\overrightarrow{BC}$ starting at B so that $C'' = C$ (as $BC \cong B'C'$ by hypothesis, $B''C'' \cong B'C'$ with $B'' = B$ by construction, and congruent segments are unique on a ray, given a starting point, by congruence axiom.)

Lay off a copy of $\angle C'$ at C'' to form $\angle DB''C''$ such that D is on the side of $\overleftrightarrow{BC}$ opposite A (this angle being unique by congruence axiom.)

A, C and D are collinear (AC and CD are both perpendicular to BC at C.) Further, $A * C * D$ (by construction at previous step.)

Lay off a copy of $C'A'$ on $\overrightarrow{C''D''}$ starting at C'' to form $C''A'' \cong C'A'$.

$A * C * A''$ as $A'' \in \overrightarrow{CD''} \backslash \{C\}$ (by construction at previous step.)

$\therefore \triangle A''B''C'' \cong \triangle A'B'C'$ by S.A.S. ($A''C'' \cong A'C'$, $\angle A''C''B'' \cong \angle C'$, and $B''C'' \cong B'C'$, all by construction at previous steps.)

$\therefore$ $A''B'' = A''B \cong A'B'$ (as corresponding parts of congruent triangles) so $\cong$ $A''B \cong AB$ (by hypothesis and transitivity of congruence by axiom.)

$\therefore$ $\triangle AA''B$ is isosceles with legs $AB \cong A''B$. Thus $\angle A \cong \angle BA''C$ by theorem (isosceles triangles have congruent base angles.)

Then $\triangle ABC \cong \triangle A''BC$ by A.A.S. ($\angle ACB \cong \angle A''CB$ by construction at previous step, $\angle A \cong \angle BA''C$ by previous step, and BC is a common side; alternatively, $\angle BCA \cong \angle BCA''$ by construction previous step, $\angle BA''C \cong \angle A$ by previous step, and $AB \cong A''B$ by construction at previous step.)

$\triangle A''BC = \triangle A''B''C'' \cong \triangle A'B'C'$ (by previous step) so $\triangle A'B'C' \cong \triangle ABC$ by transitivity of congruence (congruence axiom.) $\square$

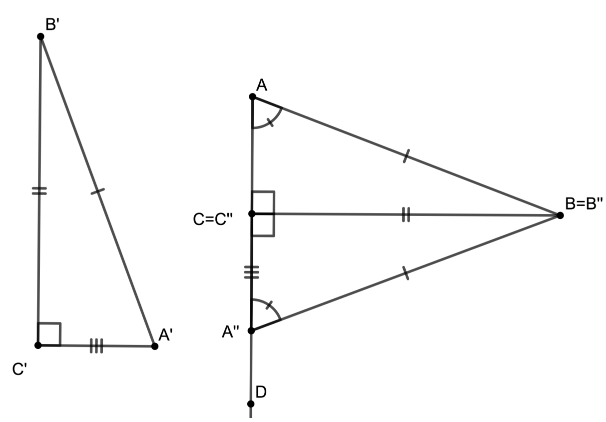


Figure 7: Construction proving HL convergence.

7.

Proposition. *Given* $\triangle ABC$ *with* M *the midpoint of* AB, $(\angle C^\circ = \angle A^\circ + \angle B^\circ) \Rightarrow (\overline{AB} = 2\overline{CM})$

Proof by construction. Given $\triangle ABC$ with ($\angle C^\circ = \angle A^\circ + \angle B^\circ$, lay off a copy of $\angle A$ at C with $\overrightarrow{CA}$ as one side and the other side, except C itself, on B's side of $\overleftrightarrow{CA}$, say $\overrightarrow{CD}$.

As $\overrightarrow{AD}$ over through C, it meets (AB) by Pasch's theorem. Say $\{E\} = \overrightarrow{AD} \cap (AB)$.

Then $\triangle ACE$ has congruent base angles $\angle A$ and $\angle ACE$ and is therefore isosceles with $AE \cong CE$ by theorem (triangle with congruent base angles has opposite sides congruent as well.)

$\angle ACE \cong \angle A$ (by construction at previous step,) $\angle ACB$ is comprised of $\angle ACE$ and $\angle BCE$ (by construction at previous step,) and $(\angle C^\circ = \angle A^\circ + \angle B^\circ$ by hypothesis, so $\angle BCE \cong \angle B$ by angle subtraction.

Then $\triangle BCE$ has congruent base angles $\angle B$ and $\angle BCE$ and is therefore isosceles with $BE \cong CE$ by theorem (ibid.)

$BE \cong CE \cong AE \Rightarrow AE \cong BE$ so E is the midpoint of AB, which has the name M by hypothesis.

Then $CM \cong AM$ and $AB = AM \cup BM$ with $AM \cong BM$ gives $\overline{AB} = 2\overline{CM}$. $\square$

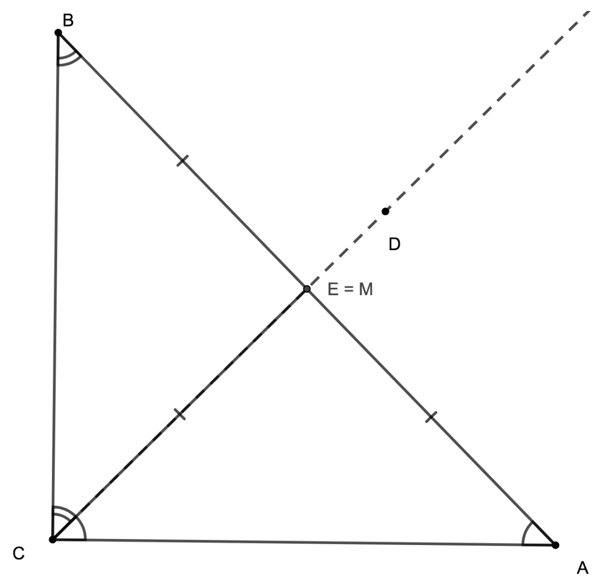


Figure 8: Triangle with largest angle whose measure is the sum of the measures of the others has a shortest median half its longest side.

8. Given convex $\square ABCD$ with $\angle A \cong \angle D$...

   (a)

   Proposition. $(AB \cong CD) \Rightarrow (\angle B \cong \angle C)$

   Proof. Draw the diagonals AC and BD – these remain within $\square ABCD$ by definition of convex.

   $\triangle ADC \cong \triangle DAB$ by S.A.S. (AD is a common side, $\angle D \cong \angle A$ by hypothesis, $AB \cong CD$ by hypothesis.)

$\therefore$ $AC \cong BD$ and $\angle ACD \cong \angle ABD$ as corresponding parts of congruent figures.

$\therefore$ $\triangle ABC \cong \triangle DCB$ by S.S.S. ($AB \cong CD$ by hypothesis, BC is a common side, $AC \cong BD$ by previous step.

$\therefore$ $\angle ACB \cong \angle CBD$ as corresponding parts of congruent figures.

$\angle ABC$ being comprised of $\angle ABD$ and $\angle CBD$, and these latter being congruent to $\angle ACB$ and $\angle ACD$, constituents of $\angle BCD$, $\angle ABC \cong \angle BCD$ by angle addition. $\square$

(b)

Proposition. $(\angle B \cong \angle C) \Rightarrow (AB \cong CD)$

Proof by construction: Construct p, the perpendicular bisector of AD. p meets (AD) by definition (at its midpoint, say F) and meets the interior of the opposite side, (BC), as $\square ABCD$ is convex (via Pasch's theorem on $\triangle BFC$, as p goes through F.) Call $p \cap BC = \{E\}$.

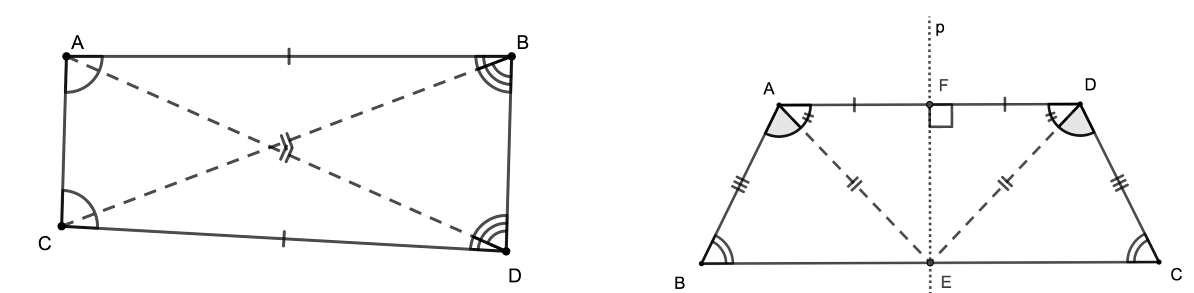
$\triangle AEF \cong \triangle DEF$ by H.L. ($AF \cong DF$ as F is the midpoint of AD by construction at previous step; $\angle AFE$ is a right angle by construction at previous step so $\angle DFE$ is as well, by definition of right angle, since $\angle DFE$ supplements right $\angle AFE$; and EF is a common leg.)

$\therefore$ $AE \cong DE$ and $\angle EAD \cong \angle EDA$ as corresponding parts of congruent figures.

Since $\angle A = \angle BAD \cong \angle ADC = \angle D$ by hypothesis, and since $\angle A$ consists of $\angle BAE$ and $\angle EAD$ whereas $\angle D$ consists of $\angle ADE$ and $\angle CDE$, with one constituent each congruent by previous step, $\angle BAE \cong \angle CDE$ by angle substraction.

$\therefore$ $\triangle ABE \cong \triangle DCE$ by A.A.S. ($\angle B \cong \angle C$ by hypothesis, $\angle BAE \cong \angle CDE$ by previous step, and Æ $\cong DE$ by previous step.)

$\therefore$ $AB \cong CD$ as corresponding parts of congruent figures. $\square$



(a) ...pair of congruent adjacent angles and opposite sides.

(b) ...two pairs of congruent adjacent angles.

Figure 9: Convex quadrilaterals with...

9.

Proposition. *Given $\triangle ABC$ with CM a median,* $[\delta(\triangle ACM) = 2\delta(\triangle ABC)] \Rightarrow [\sum(\triangle ABC) = \sum(\triangle AMC) = \sum(\triangle BCM) = 180°]$.

Proof. By construction, $\triangle ACM$ is within $\triangle ABC$.

By previous theorem, defect is additive, so that $\delta(\triangle ACM) \leq \delta(\triangle ABC)$ (as the latter is within the former by previous step.)

$\delta(\triangle) \geq 0°$ (Saccheri-Legendre theorem) so that $\delta(\triangle ACM) = 2\delta(\triangle ABC) \Rightarrow \delta(\triangle ACM) \geq \delta(\triangle ABC)$.

$\{[\delta(\triangle ACM) \geq \delta(\triangle ABC)] \wedge [\delta(\triangle ACM) \leq \delta(\triangle ABC)]\} \Rightarrow [\delta(\triangle ACM) = 0°]$.

$\therefore \forall \triangle, \delta(\triangle) 0°$ by previous theorem (Legendre's.)

$\therefore \forall \triangle, \sum(\triangle) = 180°$ by definition of defect ($\delta(\triangle) = 180° - \sum(\triangle)$.) □

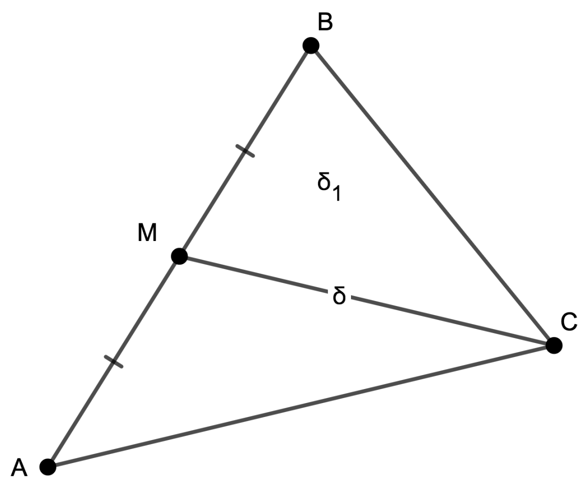


Figure 10: Inner triangle has same defect at most.

10. (a)

Proposition. *Given point D interior to $\triangle ABC$, $\overline{AB} + \overline{BC} > \overline{AD} + \overline{CD}$.*

Proof. Consider $\overrightarrow{AD}$. Since D is interior to $\triangle ABC$ and A is a vertex, $\overrightarrow{AD}$ meets (BC) by Pasch's theorem. Say $\{E\} = \overrightarrow{AD} \cap BC$.
Then $\triangle ACE$ is a triangle, so $\overline{AC} + \overline{CE} > \overline{AE}$ by the triangle inequality.
Likewise, $\triangle BDE$ is a triangle, so $\overline{BE} + \overline{DE} > \overline{BD}$.
Adding these inequalities yields: $\overline{AC} + \overline{CE} + \overline{BE} + \overline{DE} > \overline{AE} + \overline{BD}$.
But then $\overline{BE} + \overline{CE} = \overline{BC}$, so we have: $\overline{AC} + \overline{BC} + \overline{DE} > \overline{AE} + \overline{BD}$.
Subtracting $\overline{DE}$ from both sides yields: $\overline{AC} + \overline{BC} > \overline{AE} - \overline{DE} + \overline{BD}$
However, $\overline{AE} = \overline{AD} + \overline{DE}$, so we have $\overline{AC} + \overline{BC} > \overline{AD} + \overline{DE} - \overline{DE} + \overline{BD} = \overline{AD} + \overline{BD}$. □

(b)

Proposition. *Given point D interior to $\triangle ABC$, $\angle ADC > \angle ABC$.*

Proof. Consider $\overrightarrow{BD}$. Since D is interior to $\triangle ABC$ and B is a vertex, $\overrightarrow{BD}$ meets (AC) by Pasch's theorem. Say $\{F\} = \overrightarrow{BD} \cap AC$.
Then $\angle ADF$ is an exterior angle of $\triangle ABD$ at D, so it is greater than the remote interior angle $\angle ABF$ by the exterior angle theorem.
Similarly, $\angle CDF > \angle CBF$.
As $\overrightarrow{DA} * \overrightarrow{DF} * (\overrightarrow{DC}$ and $\overrightarrow{BA} * \overrightarrow{BF} * \overrightarrow{BC}$, $\angle ADC > \angle ABC$ by angle addition, given the previous two steps. □

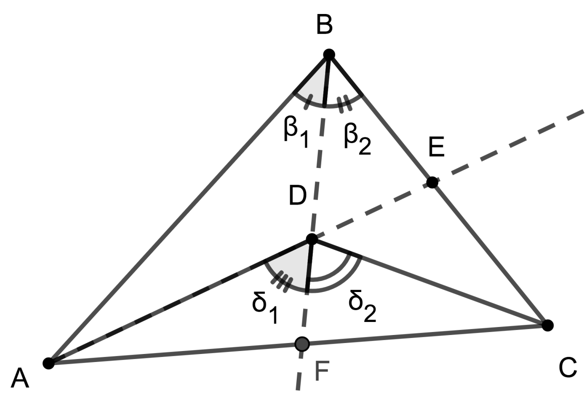


Figure 11: Inner triangle on same base has smaller perimeter and larger apex angle.

(c)

Proposition. *Given $\triangle ABC$, $(D, E \in (AC)$ s.t. $AD \cong CE) \Rightarrow (\overline{AB} + \overline{AC} > \overline{AD} + \overline{AE})$.*

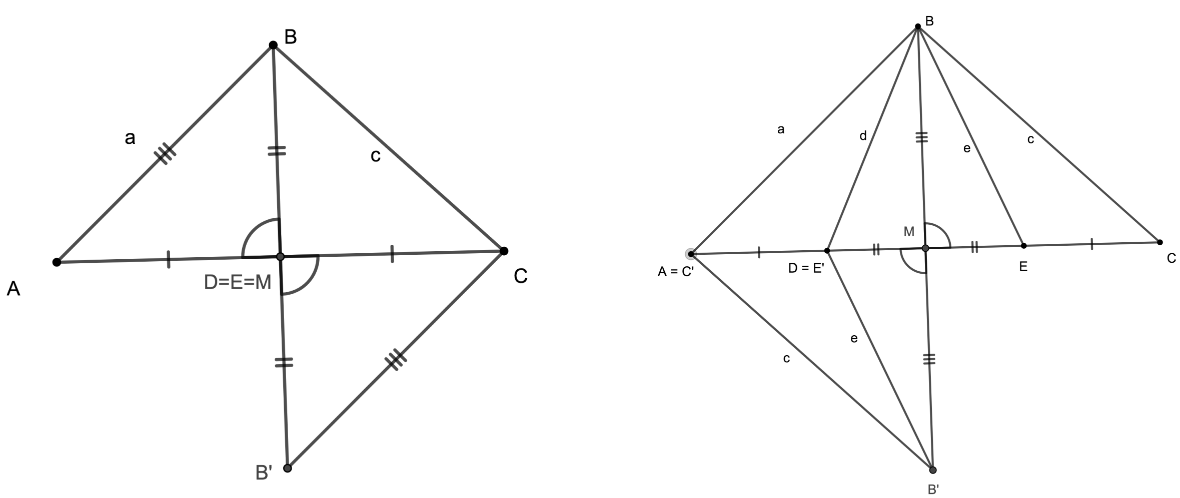
Proof. Given $\triangle ABC$ with $(D, E \in (AC)$ s.t. $AD \cong CE$, construct $M \in (AC)$, the midpoint of AC.
By segment subtraction, $AD \cong CE \Rightarrow AE \cong CD$ (they differ by the common segment DE.)
Thus exactly one of $D = E = M$ or $D \neq E$ and M is the midpoint of DE as well.

Case 3 ($D = E = M$): Construct B' by doubling BM.
Connect B' to C produces $\triangle B'MC$.
$\angle B'MC \cong \angle AMB$ as vertical angles.
$AM \cong CM$ by definition of midpoint.
$\therefore$ $\triangle ABM \cong \triangle CB'M$ by S.A.S. ($AM \cong CM$ by construction at previous step, $\angle AMB \cong \angle B'MC$ by previous step, $BM \cong B'M$ by construction at previous step.)

$\therefore AB \cong B'C$ (as corresponding parts of congruent figures.)
Applying the triangle inequality to $\triangle BB'C$ yields: $\overline{BC} + \overline{B'C} > \overline{BB'}$.
By construction, $BB' = 2BM \Rightarrow \overline{BB'} = 2\overline{BM} = \overline{BD} + \overline{BE}$ ($BM = BD = BE$ by hypothesis.)
$B'C \cong AB$ (previous step,) so $\overline{B'C} = \overline{AB}$.
$\therefore \overline{AB} + \overline{BC} > \overline{BD} + \overline{BE}$

Case 4 ($D \neq E$; $DM \cong EM$): Construct B' by doubling BM.
Connect B' to D producing $\triangle B'DM$.
$\angle AMB' \cong \angle CMB$ as vertical angles.
$DM \cong EM$ by construction (M is the midpoint of DE.)
$BM \cong B'M$ by construction at previous step.
$\therefore \triangle BME \cong \triangle B'MD$ by S.A.S. ($DM \cong EM$, $\angle DMB' \cong \angle BME$, $BM \cong B'M$.)
$\therefore B'D \cong BE$ (congruent triangles.)
Similarly, connect B' to A to produce $\triangle AMB'$.
$AM \cong CM$ by construction (M is the midpoint of AC.)
$\therefore \triangle AMB' \cong \triangle CMB$ by S.A.S. ($AM \cong CM$, $\angle AMB' \cong \angle BMC$, $BM \cong B'M$.)
$\therefore AB' \cong BC$ (congruent triangles.)[3]
Consider $\triangle ABB'$ – it is of the same form as the triangle from 10. (a), so apply the result of that part to yield: $\overline{AB} + \overline{AB'} > \overline{BD} + \overline{B'D}$.
$AB' \cong BC$ and $B'D \cong BE$, so we have $\overline{AB} + \overline{BC} > \overline{BD} + \overline{BE}$.

$\square$



(a) Case 1 – $D = E$.

(b) Case 2 – $D \neq E$.

Figure 12: Inner triangle with same apex also has smaller perimeter.

[3]We can think of the result of the construction of these two triangles as the reflection of $\triangle BMC$ over $\overleftrightarrow{BM}$ and $\overleftrightarrow{AC}$. The naming of prime points in the figure reflects this concept.