Mike Sorice **College Geometry II**
Spring 2019 **Quiz 1**

Let $l \times m$ denote the relation line l intersects line m, i.e. $l \neq m \wedge l \nparallel m$.
Let $P \text{ I } l$ denote the relation point P is on line l, i.e. $\{P\} \cap \{l\} = \{P\}$.

1. Statement to consider: Given (distinct) lines a, b, and c, $[(a \times b) \wedge (b \times c)] \Rightarrow (a \times c)$.

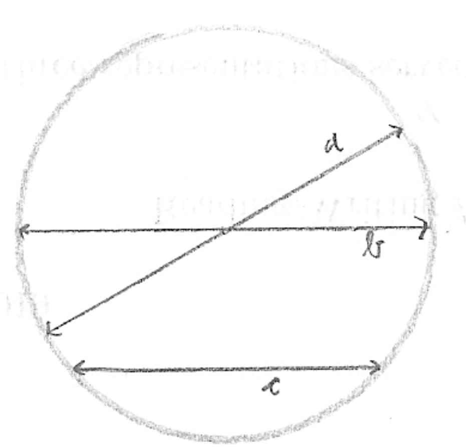
Proof in $\mathbb{E}^2$. Suppose $a \parallel c$. Then $a \times b \wedge b \times c$, but $a \not\times c$.
(∃! such c through any point $P \not\text{I}\, a$ by Euclid's 5th Postulate (E-5.))
$\therefore$ is is not the case in $\mathbb{E}^2$ that $a \times b \wedge b \times c \Rightarrow a \times c$.

Proof in $\mathbb{H}^2$. Suppose $a \parallel c$. Then $a \times b \wedge b \times c$, but $a \not\times c$.
(∃ at least two such c through any point $P \not\text{I}\, a$ by Euclid's 5th Postulate (E-5.))
$\therefore$ is is not the case in $\mathbb{H}^2$ that $a \times b \wedge b \times c \Rightarrow a \times c$.

2. Statement to consider: Given (distinct) lines a, b, and c, $[(a \times b) \wedge (b \parallel c)] \Rightarrow (a \times c)$.

Proof in $\mathbb{E}^2$. $a \times b$ by hypothesis, so $\{a\} \cap \{b\} = \{P\}$ for P some point by Incidence Axiom 1.
$P \text{ I } b \Rightarrow P \not\text{I}\, c$ as $b \parallel c$ by hypothesis.
$P \not\text{I}\, c \Rightarrow$ ∃! line through P parallel to c by Euclid's 5th Postulate.
As $P \text{ I } b$ by construction and $b \parallel c$ by hypothesis, b is that unique line.
In particular, as $a \neq b$ by hypothesis but $P \text{ I } a$ by construction, $a \nparallel c$.
$a \neq c$ by hypothesis.
$\therefore a \times c$ by definition.

Falsity in $\mathbb{H}^2$. Our proof of this statement in $\mathbb{E}^2$ depends on Euclid's 5th postulate, so it will not hold in $\mathbb{H}^2$.
Here is a counterexample in $\mathbb{K}^2 \models \mathbb{H}^2$:



3. Given an angle $\angle \Sigma A \Omega = \angle A$, for all points P interior to $\angle A$ and all lines l intersecting P, l meets either both sides of $\angle A$ or one side.

Proof in $\mathbb{E}^2$. l is distinct from both $\overleftrightarrow{A\Sigma} = s$ and $\overleftrightarrow{A\Omega} = w$ as $P \,\textsc{i}\, l$ but $P \not\!\textsc{i}\, s, w$ by definition as P is interior to $\angle A$ by hypothesis.

Suppose $l \parallel s$.

Then l does not meet s by definition, so all points of l are on the same side of s, namely P's (and Ω's) side.

However, since $s \times w$ (at P), therefore $l \times w$ by the result of 2(a).

Further, since all points of l are on Ω's side of s, l meets a point of w on Ω's side of A, i.e. a point of $\overrightarrow{A\Omega}$ by definition, which is a side of $\angle A$.

Suppose $l \parallel w$.

Then, similarly, l meets $\overrightarrow{A\Sigma}$.

Finally, suppose $l \nparallel s \wedge l \nparallel w$.

Then by definition $l \times s \wedge l \times w$. Let $\{l\} \cap \{s\} = \{S\}$ and $\{l\} \cap \{w\} = \{W\}$.

By Betweenness Axiom 2, exactly one of $S = W$, $S * P * W, P * S * W$, or $P * W * S$.

If $S = W$, then $S = W = A$ since $\{s\} \cap \{w\} = \{A\}$. A being the vertex of both $\overrightarrow{A\Sigma}$ and $\overrightarrow{A\Omega}$, l therefore meets both sides of $\angle A$.

If $S * P * W$, then $\overrightarrow{PS}$ meets $\overrightarrow{A\Sigma}$ since all non-S points on PS are on P's (and Ω's) side of s. Similarly, $\overrightarrow{PW}$ meets $\overrightarrow{A\Omega}$. Therefore, l meets both sides of $\angle A$.

If $P * S * W$, then $\overrightarrow{PS}$ meets $\overrightarrow{A\Sigma}$ since all non-S points on PS are on P's (and Ω's) side of s. Therefore l meets one side of $\angle A$.

Similarly, if $P * W * S$, then $\overrightarrow{PW}$ meets $\overrightarrow{A\Omega}$ so l meets one side of $\angle A$.

$\square$

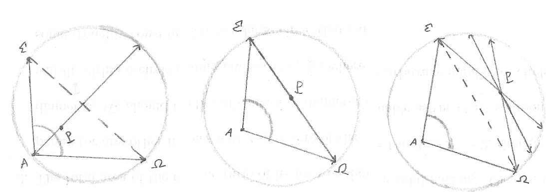
Falsity in $\mathbb{H}^2$. Once again, our proof of this statement in $\mathbb{E}^2$ depends on Euclid's 5th postulate (through its use of the result of 2(b),) so it will not hold in $\mathbb{H}^2$.

In fact, there are two distinct ways this can seen to fail in $\mathbb{K}^2 \models \mathbb{H}^2$.

First, if A is an ideal point, then any P inside $\angle A$ has at least one line through it, $\overleftrightarrow{PA}$, that is parallel to both sides of the angle.

Second (whether A is an ordinary or ideal point,) any P not on A's side of $\overleftrightarrow{\Sigma\Omega}$ has at least one line parallel to both sides of the angle – exactly one if $P \,\textsc{i}\, \overleftrightarrow{\Sigma\Omega}$ and arbitrarily many if $P \not\!\textsc{i}\, \overleftrightarrow{\Sigma\Omega}$.

These cases are illustrated below:



In summary, the set of points within $\angle A$ where the theorem can fail are: all points if A is an ideal point, all points on $\overleftrightarrow{\Sigma\Omega}$, and all points on the side of $\overleftrightarrow{\Sigma\Omega}$ opposite A.

We can therefore state a more restricted theorem that is true in $\mathbb{K}^2$ as follows: Given an angle $\angle\Sigma A\Omega = \angle A$, with Σ and Ω ideal points but A not an ideal point, for all points P interior to $\angle A$ and on A's side of $\overleftrightarrow{\Sigma\Omega}$ and all lines l intersecting P, l meets at least one side of $\angle A$.