Mike Sorice
Spring 2019

College Geometry II
Homework -4

We wish to consider whether the following theorem of $\mathbb{E}^2$ is true in $\mathbb{N}^2$: *Two angles with mutually perpendicular sides are equal if they are both acute, right, or obtuse; or their sum is* $180°$ *if one is acute while the other is obtuse.*

As this is known to hold in $\mathbb{E}^2$, it remains only to consider whether it holds in $\mathbb{H}^2$. There are four cases to consider:

Case 1 (Both acute):

Proposition. *Two acute angles with mutually perpendicular sides are not necessary congruent in* $\mathbb{H}^2$.

Proof. Given two acute angles with mutually perpendicular sides, $\angle CAE$ and $\angle CBD$, with $\angle D$ and $\angle E$ the right angles between sides, select a point D' such that $C * D * D'$ and erect a perpendicular.

Label as B' the point of intersection between this perpendicular and $\overleftrightarrow{CB}$.

Now both $\angle B'$ and $\angle CBD$ have sides mutually perpendicular to those of $\angle A$, so if the theorem holds, $\angle B' \cong \angle A \cong \angle CBD$.

Let $\angle CBD^\circ = \beta$, $\angle CB'D'^\circ = \beta'$, and $\angle BCD^\circ = \gamma$.

However $\triangle BCD \subset \triangle B'CD'$ so, in $\mathbb{H}^2$:

$$\sum(\triangle BCD) > \sum(\triangle B'CD')$$
$$\beta + \gamma + e > \beta' + \gamma + e$$
$$\therefore \beta > \beta' \Rightarrow \beta \neq \beta'$$
$$\therefore \angle CBD \ncong \angle B'.$$

□

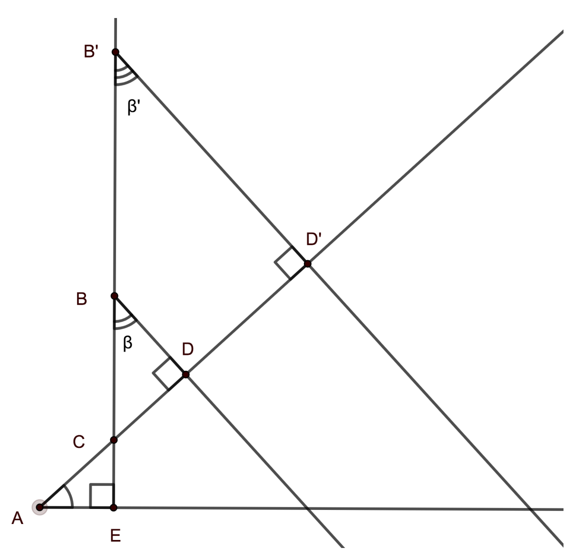

Figure 1: Two acute angles with mutually perpendicular sides are not necessarily congruent in $\mathbb{H}^2$.

Case 2 (Both right):

Proposition. *Two right angles never have mutually perpendicular sides in* $\mathbb{H}^2$.

Proof. Let $\angle CAD$ and $\angle CBD$ be two right angles with mutually perpendicular sides $\angle C$ and $\angle D$ being the right angles between sides.

But then $\square ACBD$ is a rectangle, which is impossible in $\mathbb{H}^2$! $\Rightarrow\!\Leftarrow$ $\square$

However, this case may be considered vacuously true, it being the case that all right angles are congruent.

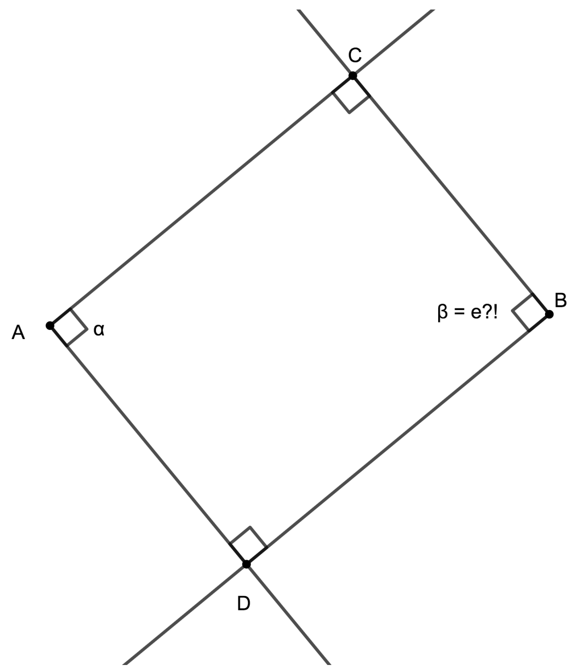

Figure 2: Two right angles with mutually perpendicular sides, which is not possible in $\mathbb{H}^2$.

Case 3 (Both obtuse):

Proposition. *Two obtuse angles with mutually perpendicular sides are not necessary congruent in* $\mathbb{H}^2$.

Proof. Given two obtuse angles with mutually perpendicular sides, $\angle A$ and $\angle CBE$, with $\angle D$ and $\angle F$ the right angles between sides, select a point D' such that $C * D * D'$ and erect a perpendicular.

Label as B' the point of intersection between this perpendicular and $\overleftrightarrow{CB}$ and select a point E' such that $D' * B' * E'$.

Now $\angle CBE$ and $\angle CB'E'$ are both angles with sides mutually perpendicular to those of $\angle A$. Therefore, if the $\mathbb{E}^2$ theorem holds, $\angle CBE \cong \angle A \cong \angle CB'E'$.

However, $\angle CBD$ and $\angle CB'D$ are acute angles with sides mutually perpendicular to those of $\angle A$ and are therefore incongruent in $\mathbb{H}^2$ by previous case.

Further, $\angle CBD$ supplements $\angle CBE$ and $\angle CB'D'$ supplements $\angle CB'E'$. As the supplements of incongruent angles are also incongruent by contrapositive of previous theorem:

$$\angle CBE \ncong \angle CB'E',$$

contradicting the theorem. $\square$

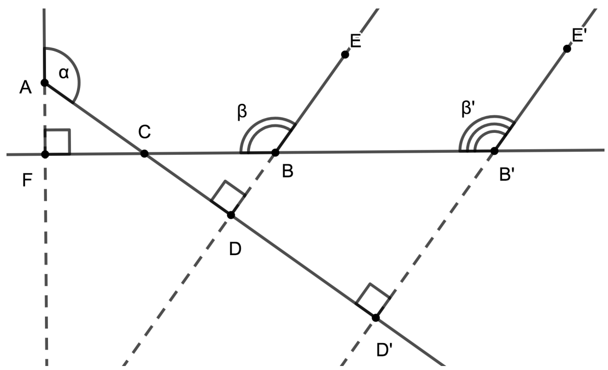

Figure 3: Two obtuse angles with mutually perpendicular sides are not necessarily congruent in $\mathbb{H}^2$.

Case 4 (One acute, one obtuse):

Proposition. *An acute and an obtuse angle with mutually perpendicular sides sum to less than* $2e$ *in* $\mathbb{H}^2$.

Proof. Let $\angle CAD$ and $\angle CBD$ be an acute and an obtuse angle with mutually perpendicular sides $\angle C$ and $\angle D$ being the right angles between sides.

By previous theorem, $\sum(\square ACBD) = \sum(\triangle ABC) + \sum(\triangle ABD) < 4e$.

Let $\angle CAD^\circ = \alpha$ and $\angle CBD^\circ = \beta$. Then:

$$\sum(\square ACBD) = \alpha + e + \beta + e < 4e \Rightarrow \alpha + \beta < 2e.$$

$\square$

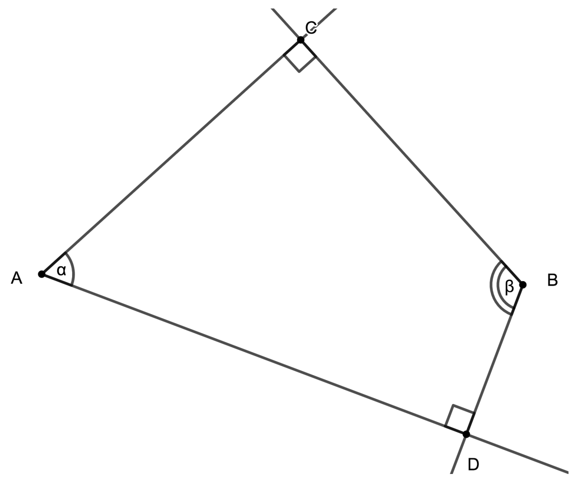

Figure 4: An obtuse and an acute angle with mutually perpendicular sides sum to less than a straight angle in $\mathbb{H}^2$.

Therefore, no case of this theorem properly holds in $\mathbb{H}^2$ or, therefore, in $\mathbb{N}^2$, though the "two right angles" case may be said to be vacuously true.