Mike Sorice — College Geometry II
Spring 2019 — Homework -1

Ch. 4, Ex. 17 (a) Suppose $O \notin AB$.

$AM \cong BM$ by definition of midpoint.
$AO \cong BO$ by definition of circle (both are radii of γ.)
$OM \cong OM$ by Congruence Axiom 2.
$\triangle AMO \cong \triangle BMO$ by Proposition 3.22 (S.S.S. congruence.)
Therefore, $\angle AMO \cong \angle BMO$ as corresponding parts of congruent triangles.
$\angle AMO$ supplements $\angle BMO$ by definition.
$\therefore$ $\angle AMO$ and $\angle BMO$ are right angles by definition. $\square$

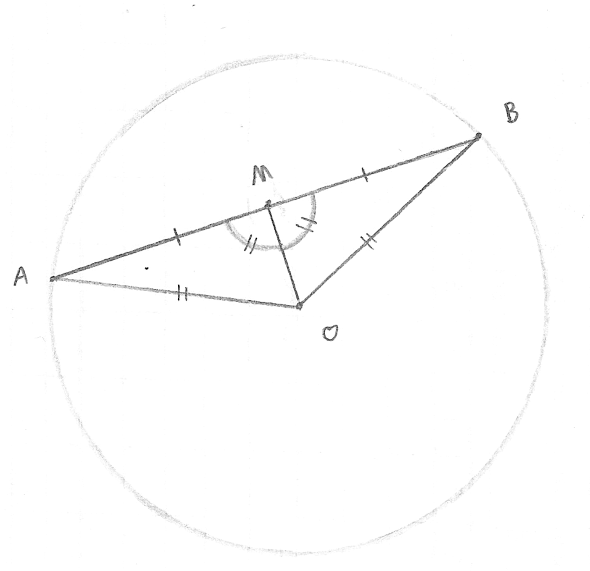

Figure 1: Radial segment through midpoint is perpendicular to (non-diameter) chord.

(b) **Case 1 (AB is not a diameter of γ):** Suppose $O \notin AB$ as in part (a).
Then $MO \perp AB$ by the result of part (a), so $\overleftrightarrow{MO} \perp AB$ as well.
Further, as M is the midpoint of AB by construction, $\overleftrightarrow{MO}$ bisects AB by definition.
Therefore $\overleftrightarrow{MO}$ is a perpendicular bisector of AB. Then $\overleftrightarrow{MO}$ is the unique perpendicular bisector of AB by Proposition 4.4 (b).
As $O \in \overleftrightarrow{MO}$, the perpendicular bisector of AB passes through O. //

Case 2 (AB is a diameter of γ): Suppose on the other hand that $O \in AB$.
Then $AO \cong BO$ by definition of circle as both are radii of γ.
Therefore O is the midpoint of AB by definition.

As the perpendicular bisector of AB bisects AB, it must therefore pass through its midpoint, O. //

Therefore, in either case, the perpendicular bisector of any chord must pass through the center of its circle. $\square$

Ch. 4, Ex. 18 Proof of Thales' theorem in Euclidean geometry Let $\circ ABC$ be a circle with $\angle BAC$ the inscribed angle of a semi-circle of $\circ ABC$.

Then BC is a diameter of $\circ ABC$ by definition of inscribed angle of semicircle.

Therefore $AO \cong BO \cong CO$ as all are radii of $\circ ABC$.

Then $\triangle ABO$ is isosceles, so $\angle ABO \cong \angle BAO$ by Proposition 3.10 (the base angles of an isosceles triangle are congruent.) Let $\angle ABO^\circ = \angle BAO^\circ = \alpha$.

Similarly, $\angle ACO \cong \angle CAO$. Let $\angle ACO^\circ = \angle CAO^\circ = \beta$.

In Euclidean geometry, Euclid's fifth postulate holds so that the sum of the measures of the angles of any triangle is 180° (by Theorem 4.5 and Proposition 4.11.)

In particular, $\alpha + \alpha + \angle AOB^\circ = 180^\circ$ and $\beta + \beta + \angle AOC^\circ = 180^\circ$. Adding these two equations yields $2\alpha + 2\beta + \angle AOB^\circ + \angle AOC^\circ = 360^\circ \Rightarrow \alpha + \beta = 180^\circ - \frac{\angle AOB^\circ + \angle AOC^\circ}{2}$.

$\angle AOB$ supplements $\angle AOC$ by definition, so $\angle AOB^\circ + \angle AOC^\circ = 180^\circ$ by Theorem 4.3 (5) (measures of supplementary angles sum to 180°.)

Therefore $\alpha + \beta = 180^\circ - \frac{\angle AOB^\circ + \angle AOC^\circ}{2} = 180^\circ - \frac{180^\circ}{2} = 90^\circ$.

As $\alpha = \angle BAO^\circ$ and $\beta = \angle CAO^\circ, \alpha + \beta = \angle BAC^\circ = 90^\circ$.

$\therefore \angle BAC$ is a right angle by Theorem 4.3 (1) (an angle is right if and only if it measures 90°.) $\square$

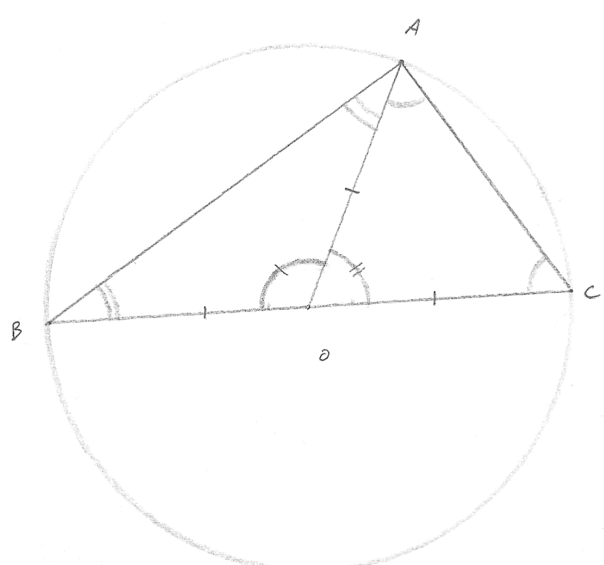

Figure 2: Inscribed $\angle BAC$ is a right angle by Thales' theorem.

Construction of non-defective triangle given Thales' theorem Given a circle γ with center O, suppose that Thales' theorem holds.

Construct two distinct diameters of γ, AC and BD.

$AO \cong BO \cong CO \cong DO$ as all are radii of γ.

$\angle ABO \cong \angle COD$ as vertical angles, so $\triangle AOB \cong \triangle COD$ by S.A.S.

$\triangle AOB$ is isosceles ($AO \cong BO$), so $\angle ABO \cong \angle BAO$.

Therefore $\angle CDO \cong \angle ABO \cong \angle BAO \cong \angle CDO$ as corresponding angles of congruent triangles. Call the common measure these angles α.

$\angle AOD \cong \angle BOC$ as vertical angles, so $\triangle AOD \cong \triangle BOC$.

$\triangle AOD$ is isosceles ($AO \cong BO$), so $\angle ADO \cong \angle DAO$.

Therefore, $\angle BCO \cong \angle ADO \cong \angle DAO \cong CBO$ as corresponding angles of congruent triangles. Call the common measure of these angles β.

Now $\angle ABC$ [1] is the vertex angle of $\triangle ABC$ whose opposite side, AC, is a diameter of γ, so $\angle ABC$ is the inscribed angle of a semi-circle of γ by definition and therefore measures $90°$ by Thales' theorem.

$\angle ABC° = \angle ABO° + \angle CBO° = \alpha + \beta$, so $\alpha + \beta = 90°$.

The sum of the measure of the angles in $\triangle ABC$ are $\angle ABC° + \angle BAO° + \angle BCO° = (\alpha + \beta) + \alpha + \beta = 2(\alpha + \beta) = 2 \times 90° = 180°$.

Therefore, the defect of $\triangle ABC$ is 0 by definition. $\square$

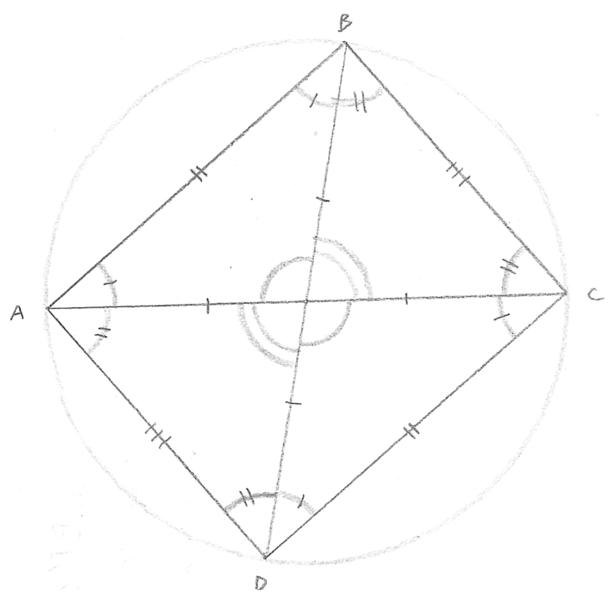

Figure 3: Pairs of defect-free triangles. Each pair of inscribed angles forms a right angle by Thales' theorem.

Proof of hinge theorem Bisector version Given $\angle ABC > \angle A'BC$ with $AB \cong A'B$, we want to show $AC > A'C$.

[1]We could equally have chosen any of the other large inscribed angles, namely $\angle BAD$, $\angle BCD$, or $\angle ADC$, as all are right angles by Thales' theorem.

Without loss of generality, assume A' is on A's side of $\overleftrightarrow{BC}$ (as otherwise we can construct A'' on A's side of $\overleftrightarrow{BC}$ s.t. $\angle A'BC \cong \angle A''BC$ and $A'C \cong A''C$ by Congruence Axioms 1 and 4, then prove $AC > A''C \cong A'C$ in the same fashion.)

$\overrightarrow{BA} * \overrightarrow{BA'} * \overrightarrow{BC}$ by definition of $<$ for angles.

By the crossbar theorem, $\overrightarrow{BA'}$ meets AC at some point, D say, such that $A * D * C$.

$\exists!$ ray r such that $\overrightarrow{BA} * r * \overrightarrow{BA'}$ and r bisects $\angle ABA'$ by Proposition 4.4(a) (each angle has a unique bisector.)

By the crossbar theorem, r meets AD at a point, say E, such that $A * E * D$.

Then by the definition of bisector, $\angle ABE \cong \angle A'BE$.

$\triangle ABE \cong \triangle A'BE$ by S.A.S. as $AB \cong A'B$ by hypothesis, $\angle ABE \cong \angle A'BE$ by definition of bisector ($E \in r$ by construction,) and $BE \cong BE$ by Congruence Axiom 2.

Therefore $AE \cong A'E$ as corresponding parts of congruent figures.

As $\triangle A'EC$ is a triangle, $\overline{A'E} + \overline{CE} > \overline{A'C}$ by Corollary 2 to Theorem 4.3 (triangle inequality)

$A * E * D \wedge A * D * C \Rightarrow A * E * C$ by Proposition 3.3.

Therefore $\overline{AC} = \overline{AE} + \overline{CE}$ by Theorem 4.3 (9) so that $\overline{AE} = \overline{AC} - \overline{CE}$.

Then, as $AE \cong A'E, \overline{AE} = \overline{A'E}$ by Theorem 4.3 (8) so that $\overline{A'E} = \overline{AC} - \overline{CE}$.

Thus, the above inequality is $\overline{A'E} + \overline{CE} = \overline{AC} - \overline{CE} + \overline{CE} = \overline{AC} > \overline{A'C}$.

$\therefore AC > A'C$ by Theorem 4.3(10). *Q.E.D.*

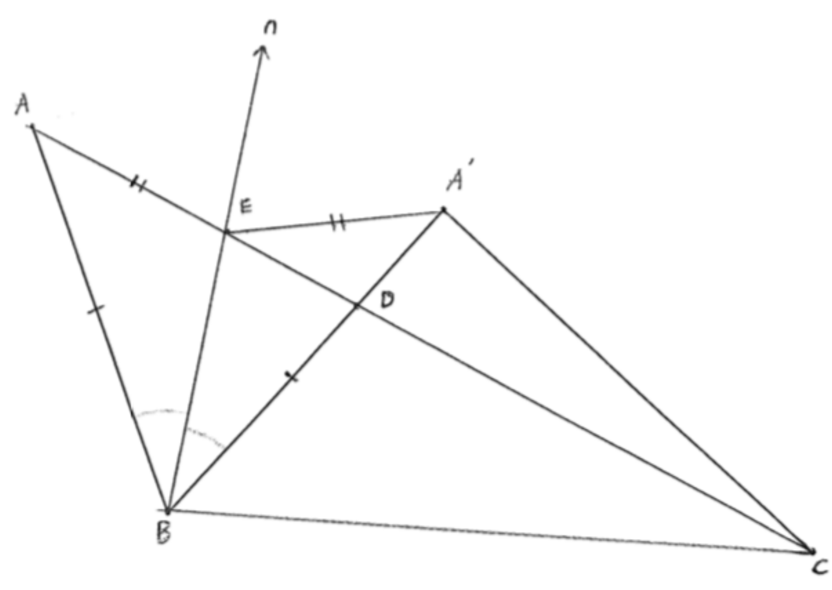

Figure 4: Hinge theorem proved using the bisector r.

Tie line version Given $\angle ABC > \angle DEF$ with $AB \cong DE \wedge BC \cong EF$, we wish to show that $AC > DF$.

Construct $\overrightarrow{BF'}$ such that F' is interior to $\angle ABC$, $BF' \cong EF$, and $\angle ABF' \cong \angle DEF$. (Such a ray as $\overrightarrow{BF'}$ exists by the definition of $<$ for angles; such a point as F' exists by Congruence Axiom 1.)

$\triangle ABF' \cong \triangle DEF$ by S.A.S. Therefore, $AF' \cong DF$ as corresponding parts of congruent triangles, so we can show that $AC > AF'$.

Further, $\triangle BCF'$ is isosceles by definition as $BC \cong BF'$ by construction, so $\angle BF'C \cong \angle BCF'$ by Proposition 3.10 (base angles of isosceles triangle are congruent.)

F' is interior to $\angle ABC$ by construction, so $\overrightarrow{BF'}$ meets AC by the crossbar theorem. Call the point where they meet G so that $A * G * C$.

$G \in \overrightarrow{BF'}$ by construction. $G \neq B$ as $B \notin AC$ but $B \in AC$. Therefore, by Betweenness Axiom 3, exactly one of $B * G * F', G =' F$, or $B * F' * G$.

Case 1 ($B * G * F'$): $\overrightarrow{CB} * \overrightarrow{CA} * \overrightarrow{CF}$ so that $\angle BCF' > \angle ACF'$ by definition.
$\angle BCF' \cong \angle BF'C$, so $\angle BF'C > \angle ACF'$ as well.
$\angle AF'C > \angle BF'C$ by definition, as $\overrightarrow{F'A} * \overrightarrow{F'B} * \overrightarrow{F'C}$ by construction ($A * G * C$.)
Therefore, *a fortiori* $\angle AF'C > \angle ACF'$.
In $\triangle ACF'$, AC is the side opposite $\angle AF'C$ and AF' that opposite $\angle ACF'$. Therefore $AC > AF'$ by Proposition 4.5 (greater side of a triangle is opposite greater angle.) //

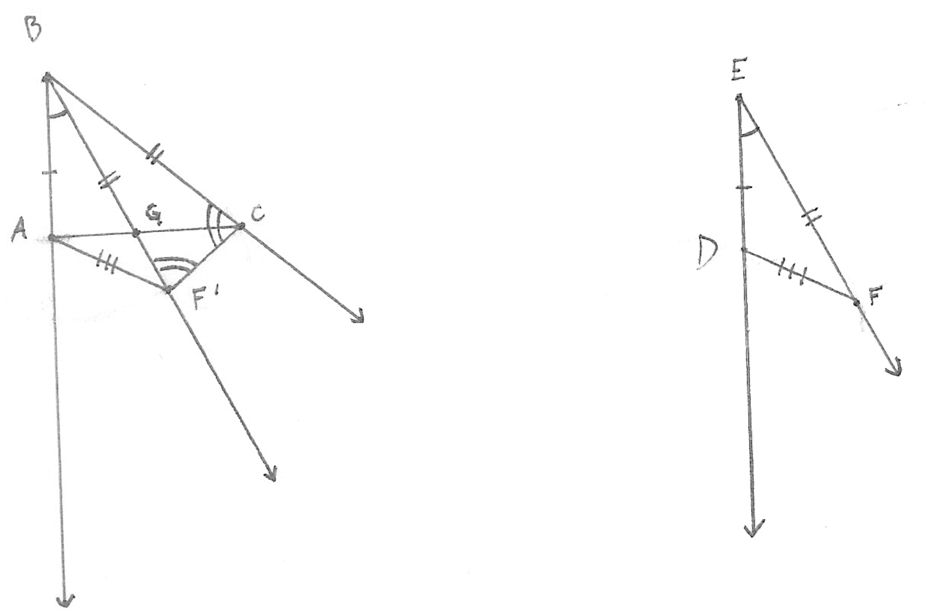

Figure 5: Hinge theorem proved using the tie line AF', case 1 – $B * G * F'$.

Case 2 ($G = F'$): $A * G * C \Rightarrow A * F' * G$ so $AG > AF'$ by definition of $>$ for segments. //

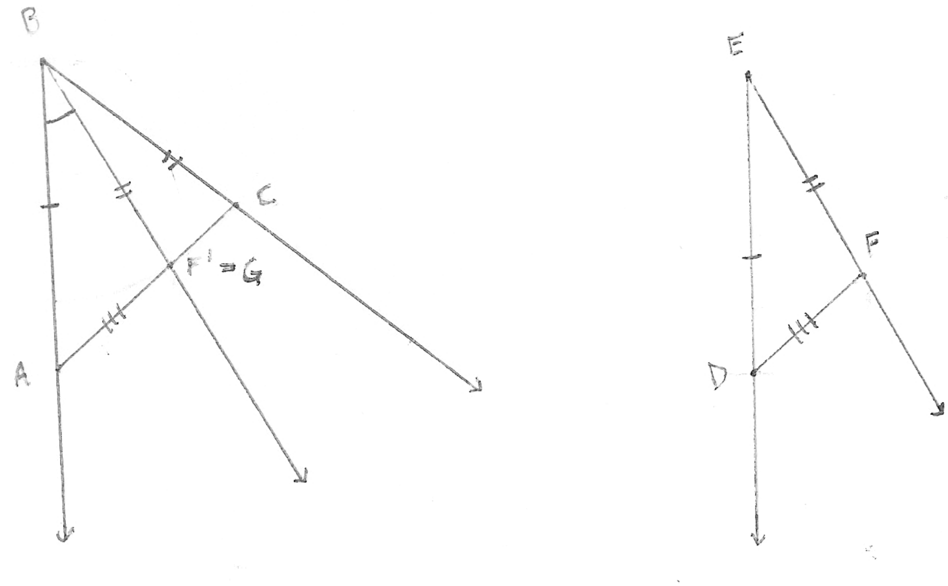

Figure 6: Hinge theorem proved using the tie line AF', case 2 – $F' = G$.

Case 3 ($B * F' * G$): $\angle CF'G$ supplements $\angle BF'C$ by definition.

Construct H such that $B * C * H$. Then $\angle F'CH$ supplements $\angle BCF'$ by definition.

$\therefore$ $\angle CF'G \cong \angle F'CH$ by Proposition 3.14 (supplements of congruent angles are congruent.)

$\overrightarrow{CF'} * \overrightarrow{CG} * \overrightarrow{CH}$ by construction $(B * F' * G \wedge B * C * H)$ so $\angle F'CG < \angle F'CH$

$\angle CF'G \cong \angle F'CH$ so $\angle CF'G > \angle F'CG$ as well.

$\overrightarrow{F'A} * \overrightarrow{F'G} * \overrightarrow{F'C}$ by construction $(A * G * C)$ so $\angle AF'C > \angle CF'G$.

Therefore, *a fortiori* $\angle AF'C > \angle F'CG$.

In $\triangle ACF'$, AC is the side opposite $\angle AF'C$ and AF' that opposite $\angle ACF'$. Therefore $AC > AF'$ by Proposition 4.5 (greater side of triangle is opposite greater angle.) //

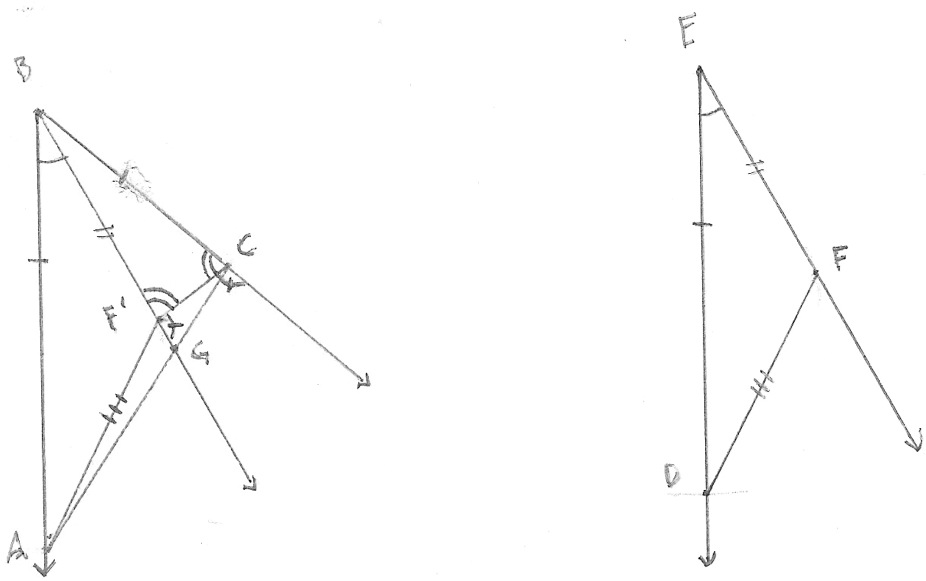

Figure 7: Hinge theorem proved using the tie line AF', case 3 – $B * F' * G$.

Therefore, in all cases, $AC > AF' \cong DF$. *Q.E.D.*