Mike Sorice **College Geometry II**
Spring 2019 **Triangle Centers and Ceva's Theorem**

Proposition 1. *The medians of a triangle are concurrent.*

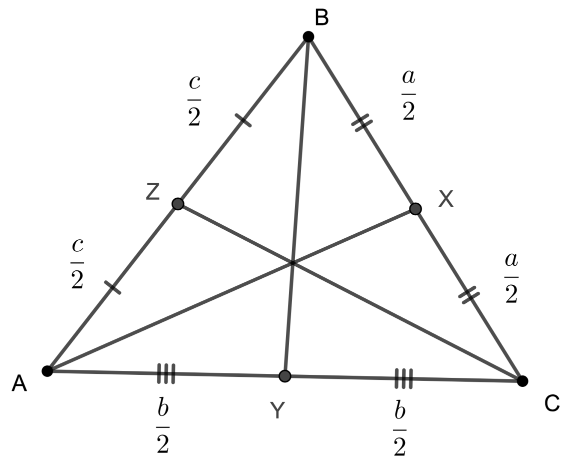


Figure 1: The medians of $\triangle ABC$ are concurrent.

Proof. Suppose $\triangle ABC$ has medians AX, BY, and CZ.

By definition of median, $|AZ| = |ZB| = \frac{c}{2}$, $|BX| = |XC| = \frac{a}{2}$, and $|CY| = |YA| = \frac{b}{2}$.

Therefore, $\frac{|AZ|}{|ZB|}\frac{|BX|}{|XC|}\frac{|CY|}{|YA|} = \frac{c/2}{c/2}\frac{a/2}{a/2}\frac{b/2}{b/2} = 1 \times 1 \times 1 = 1$.

Consequently, by the converse of Ceva's theorem, AX, BY, and CZ meet at a single point. $\square$

Proposition 2. *The altitudes of a triangle meet at a single point.*

Proof. Suppose $\triangle ABC$ has altitudes AX, BY, and CZ.

By definition of altitude, $\angle AXB$ and $\angle CZB$ are both right angles.

Consequently, $\triangle AXB \sim \triangle CZB$ by angle-angle similarity ($\angle AXB \cong \angle CZB$ by Euclid's IVth postulate, $\angle ABC$ is a common angle.)

By similar triangles, $\frac{|AB|}{|BC|} = \frac{|BX|}{|BZ|}$.

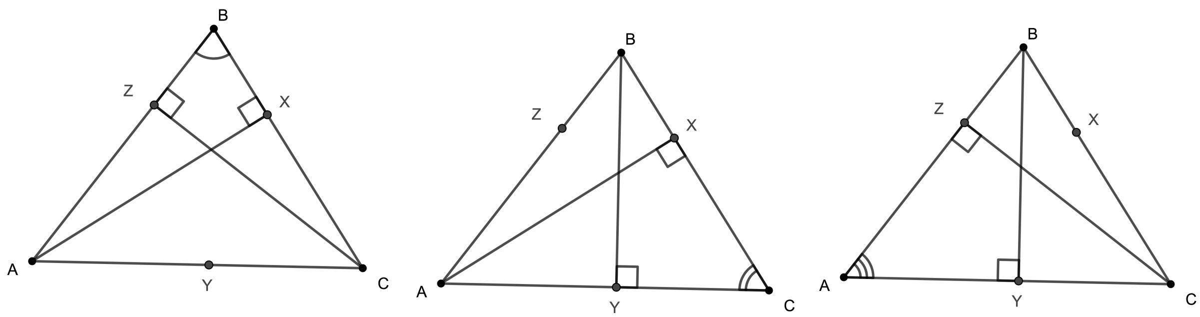
Similarly, $\angle AXC$ and $\angle BYC$ are both right, so $\triangle AXC \sim \triangle BYC$ and $\frac{|AC|}{|BC|} = \frac{|CX|}{|CY|}$.

Similarly again, $\angle BYA$ and $\angle CZA$ are right, $\triangle BYA \sim \triangle CZA$, and $\frac{|AB|}{|AC|} = \frac{|AY|}{|AZ|}$.

Consider the Cevian product of ratios of parts of sides:

$$\begin{aligned}\frac{|AZ|}{|BZ|}\frac{|BX|}{|CX|}\frac{|CY|}{|AY|} &= \left(\frac{|AZ|}{|AY|}\right)\left(\frac{|BX|}{|BZ|}\right)\left(\frac{|CY|}{|CX|}\right) = \left(\frac{|AC|}{|AB|}\right)\left(\frac{|AB|}{|BC|}\right)\left(\frac{|BC|}{|AC|}\right)\\ &= \frac{|AC|}{|AC|}\frac{|AB|}{|AB|}\frac{|BC|}{|BC|} = 1.\end{aligned}$$

Thus, by the converse of Ceva's theorem, AX, BY, and CZ meet at a single point. $\square$



(a) $\triangle ABX \sim \triangle CBX$. (b) $\triangle ACX \sim \triangle BCY$. (c) $\triangle ABY \sim \triangle ACZ$.

Figure 2: Similar triangles underlie concurrence of altitudes by Ceva's theorem.

Proposition 3. *The angle bisectors of a triangle meet at a single point.*

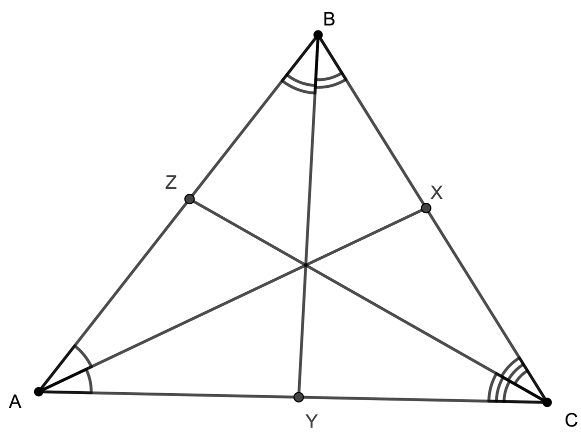


Figure 3: The angle bisectors of $\triangle ABC$ are concurrent.

Proof. Suppose $\triangle ABC$ has angle bisectors AX, BY, and CZ.

By previous theorem, $\frac{|BX|}{|CX|} = \frac{|AB|}{|AC|}$, $\frac{|AY|}{|CY|} = \frac{|AB|}{|BC|}$, and $\frac{|AZ|}{|BZ|} = \frac{|AC|}{|BC|}$.

Consider the Cevian product of ratios of parts of sides:

$$\frac{|AZ|}{|BZ|}\frac{|BX|}{|CX|}\frac{|CY|}{|AY|} = \frac{|AC|}{|BC|}\frac{|AB|}{|AC|}\frac{|BC|}{|AB|} = \frac{|AC|}{|AC|}\frac{|AB|}{|AB|}\frac{|BC|}{|BC|} = 1.$$

Therefore, by the converse of Ceva's theorem, AX, BY, and CZ meet at a single point. $\square$