Mike Sorice — College Geometry II
Spring 2019 — Homework 3

①

Proposition. *Given* $\triangle ABC$ *in* $\mathbb{E}^2$ *with medians measuring* m_a*,* m_b*, and* m_c*:*

$$\frac{1}{3}m_a + \frac{2}{3}m_b > \frac{1}{2}a.$$

Proof. Let M_A be the midpoint of BC (and therefore the foot of the median from A) and M_B the midpoint of AC (and foot of median from B.) Let G denote the intersection between the medians.

The triangle inequality on $\triangle BGM$ gives:

$$\overline{BG} + \overline{M_AG} > \overline{BM_A}.$$

By previous theorem, the medians cut each other into pieces having a $2:1$ ratio, so that:

$$\overline{BG} = \frac{2}{3}\overline{BM_B} = \frac{2}{3}m_b$$

and

$$\overline{GM_A} = \frac{1}{3}\overline{AM_A} = \frac{1}{3}m_a.$$

As M_A is the the midpoint of BC:

$$\overline{BM_A} = \frac{1}{2}a.$$

$$\therefore \frac{2}{3}m_b + \frac{1}{3}m_a > \frac{1}{2}a.$$

□

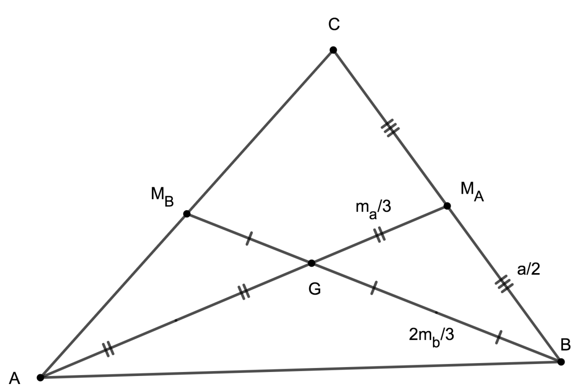

Figure 1: Third and two-third portions of medians make triangle with half side.

②

Proposition. *Given $\triangle ABC$ in $\mathbb{E}^2$ with medians measuring m_a, m_b, and m_c:*

$$m_a + m_b + m_c > \frac{3}{4}(a+b+c).$$

Proof. Let G be the intersection of the medians, a single point by previous theorem. The triangle inequality on $\triangle AGB$ gives:

$$\overline{AG} + \overline{BG} > \overline{AB} = c$$

.

By previous theorem, the medians cut each other at G into pieces having $2:1$. Therefore $\overline{AG} = \frac{2}{3}m_a$ and $\overline{BG} = \frac{2}{3}m_b$ so that:

$$\frac{2}{3}(m_a + m_b) > c.$$

Similarly, the triangle inequality on $\triangle BCG$ gives:

$$\frac{2}{3}(m_b + m_c) > a$$

and the triangle inequality on $\triangle ACG$ gives:

$$\frac{2}{3}(m_a + m_c) > b.$$

Adding these three inequalities yields:

$$\left[\frac{2}{3}(2m_a + 2m_b + 2m_c) > a+b+c\right] \Leftrightarrow \left[m_a + m_b + m_c > \frac{3}{4}(a+b+c)\right].$$

□

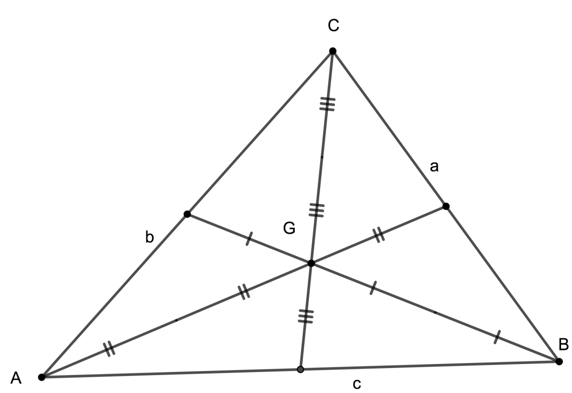

Figure 2: Two-third portions of medians from two points make triangle with side across third point.