Mike Sorice
Spring 2019

College Geometry II
Homework 2

① Let $h_{(-P)}(l)$ denote the half-plane across l from P.

Proposition. *Given $\angle \Sigma A \Omega$ with A an ordinary point of $\mathbb{K}^2$ and Σ and Ω ideal points, $\forall P \in h_{(-A)}\left(\overleftrightarrow{\Sigma\Omega}\right), \forall l$ s.t. $P \in l$, if l meets $\overrightarrow{A\Sigma}$ and l goes through no vertex of $\triangle A\Sigma\Omega$, then l does not meet $\overrightarrow{A\Omega}$.*

Proof.
1. Let P be some point of $h_{(-A)}\left(\overleftrightarrow{\Sigma\Omega}\right)$ and l some line such that $P \in l$ and l intersects $\overrightarrow{A\Sigma}$, but $(A \notin l) \wedge (\Sigma \notin l) \wedge (\Omega \notin l)$
2. All points of $\overrightarrow{A\Sigma}\backslash\{\Sigma\}$ are on $h_{(A)}\left(\overleftrightarrow{\Sigma\Omega}\right)$ as $\overleftrightarrow{A\Sigma} \cap \overleftrightarrow{\Sigma\Omega} = \Sigma$ and $\left(Q \in \overrightarrow{A\Sigma}\backslash\{\Sigma\}\right) \Rightarrow [(Q = A) \vee (A * Q * \Sigma)]$ by definition of ray.
3. Since l meets $\overrightarrow{A\Sigma} \subset \overleftrightarrow{A\Sigma}$ by hypothesis (at 1) and definition of ray, and since $l \neq \overleftrightarrow{A\Sigma}$ (as $\Sigma \notin l$ by hypothesis at 1,) l meets $\overleftrightarrow{A\Sigma}$ – and $\overrightarrow{A\Sigma}$ – at a single point by Proposition 2.1. Say $l \cap \overrightarrow{A\Sigma} = S$.
4. Then $S \in h_{(A)}\left(\overleftrightarrow{\Sigma\Omega}\right)$ (by 2, as $S \in \overrightarrow{A\Sigma}\backslash\{\Sigma\}$ by construction at 3,) so that $\exists B$ s.t. $\left(B = l \cap \overleftrightarrow{\Sigma\Omega}\right) \wedge (S * B * P)$ by plane separation. Further, $(B \neq \Sigma) \wedge (B \neq \Omega)$ (by hypothesis at 1) since $B \in l$, so $\Sigma * B * \Omega$.
5. Thus l intersects the interior of $\Sigma\Omega$, a side of $\triangle A\Sigma\Omega$ by the definition of side of a triangle.
6. Consequently, by Pasch's theorem, l meets either A or exactly one other side of $\triangle A\Sigma\Omega$. However, l does not go through A (hypothesis at 1,) so l meets exactly one of $A\Sigma$ or $A\Omega$.
7. But l meets $A\Sigma$ (hypothesis at 1,) so l does not meet $A\Omega = \overrightarrow{A\Omega}$.
8. P being an arbitrary point of $h_{(-A)}\left(\overleftrightarrow{\Sigma\Omega}\right)$ and l being an arbitrary line through P that meets $\overrightarrow{A\Sigma}$ and no vertex of $\triangle A\Sigma\Omega$, what holds for P and l holds for all such points and lines.

□

② Let $\mathcal{P}^{(n)}$ denote an n-gon and $\delta\left(\mathcal{P}^{(n)}\right) = (n-2)180^\circ - \sum\left(\mathcal{P}^{(n)}\right)$ denote the defect of $\mathcal{P}^{(n)}$.

Let $\mathcal{P}^{(n)} = \bigsqcup_{i=1}^{n-2} \triangle_i$ for some triangles $\triangle_1, \triangle_2, ..., \triangle_{n-2}$.

Proposition. $\delta\left(\mathcal{P}^{(n)}\right) = \sum_{i=1}^{n-2} \delta\left(\triangle_i\right)$.

Lemma. *All polygons can be cut into triangles.*

Proof of Lemma. Let $\mathcal{P}^{(3)}$ be a triangle.

Then $\mathcal{P}^{(3)}$ is already cut into triangles.

Suppose for induction that any polygon of at most $n-1$ vertices can be cut into triangles.

Let $\mathcal{P}^{(n)}$ have n sides and vertices.

Then drawing any diagonal of $\mathcal{P}^{(n)}$ cuts it into two polygons, $\mathcal{P}^{(n)} = \mathcal{R}^{(m)} \sqcup \mathcal{S}^{(n-m+2)}$, with $3 \leq m \leq n-1$.

But then $\mathcal{R}^{(m)}$ and $\mathcal{S}^{(n-m+2)}$ are triangulable by the inductive hypothesis, so that $\mathcal{R}^{(m)} = \bigsqcup_i \triangle_{r_i}$, $\mathcal{S}^{(n-m+2)} = \bigsqcup_j \triangle_{s_j}$, and $\mathcal{P}^{(n)} = \left(\bigsqcup_i \triangle_{r_i}\right) \sqcup \left(\bigsqcup_j \triangle_{s_j}\right)$

$\square$

Proof of Proposition. Let $\mathcal{P}^{(3)}$ be a triangle.

Then $\delta\left(\mathcal{P}^{(3)}\right) = \sum_{i=1}^{1} \delta\left(\triangle_i\right)$ identically as $\mathcal{P}^{(3)} = \triangle_1 = \bigsqcup_{i=1}^{1} \triangle_i$ by hypothesis.

Suppose for induction that $\mathcal{P}^{(n)} = \bigsqcup_{i=1}^{n-2} \triangle_i \Rightarrow \delta\left(\mathcal{P}^{(n)}\right) = \sum_{i=1}^{n-2} \delta\left(\triangle_i\right) \forall n \in \mathbb{N}$ such that $3 \leq n \leq N-1$.

Consider $\mathcal{P}^{(N)} = \bigsqcup_{i=1}^{N-2} \triangle_i$.

Any diagonal of $\mathcal{P}^{(N)}$ is also at least a connected union of sides of the $\{\triangle_i\}$ that unite to form $\mathcal{P}^{(N)}$ such that cutting along a diagonal creates two polygons such that $\mathcal{P}^{(N)} = \mathcal{R}^{(N-M+2)} \sqcup \mathcal{S}^{(M)}$ with $3 \leq M \leq N-1$ and $\mathcal{R}^{(N-M+2)}$ and $\mathcal{S}^{(M)}$ are disjoint unions of triangles from $\{\triangle_i\}$. Without loss of generality, say $\mathcal{R}^{(N-M+2)} = \bigsqcup_{i=1}^{L} \triangle_i$ and $\mathcal{S}^{(M)} = \bigsqcup_{i=L}^{N-2} \triangle_i$.

By construction, $\sum\left(\mathcal{P}^{(N)}\right) = \sum\left(\mathcal{R}^{(N-M+2)}\right) + \sum\left(\mathcal{S}^{(M)}\right)$ so that $\delta\left(\mathcal{P}^{(N)}\right) = (N-2)180° - \sum\left(\mathcal{P}^{(N)}\right) = (N-2)180° - \sum\left(\mathcal{R}^{(N-M+2)}\right) - \sum\left(\mathcal{S}^{(M)}\right) = (N+M-M-2)180° - \sum\left(\mathcal{R}^{(N-M+2)}\right) - \sum\left(\mathcal{S}^{(M)}\right) = (N-M)180° - \sum\left(\mathcal{R}^{(N-M+2)}\right) + (M-2)180° - \sum\left(\mathcal{S}^{(M)}\right) = \delta\left(\mathcal{R}^{(N-M+2)}\right) + \delta\left(\mathcal{S}^{(M)}\right)$.

By the inductive hypothesis, $\delta\left(\mathcal{R}^{(N-M+2)}\right) = \sum_{i=1}^{L} \delta\left(\triangle_i\right)$ and $\delta\left(\mathcal{S}^{(M)}\right) = \sum_{i=L}^{N-2} \delta\left(\triangle_i\right)$ so that $\delta\left(\mathcal{P}^{(N)}\right) = \sum_{i=1}^{L} \delta\left(\triangle_i\right) + \sum_{i=L}^{N-2} \delta\left(\triangle_i\right) = \sum_{i=1}^{N-2} \delta\left(\triangle_i\right)$. $\square$

③ Let $\mathcal{P} = \bigsqcup_{k=1}^{n} \mathcal{P}_k$.

Proposition. $\delta(\mathcal{P}) = \sum\limits_{k=1}^{n} \delta(\mathcal{P}_k)$.

Proof of Proposition. All polygons having a triangulation by previous lemma, say $\mathcal{P}_k = \bigsqcup\limits_{i=1}^{n_k} \triangle_{k_i}$.

$\mathcal{P}$ then has triangulation $\mathcal{P} = \bigsqcup\limits_{k=1}^{n} \bigsqcup\limits_{i=1}^{n(k)} \triangle_{k_i}$.

Thus, by the result of ②, $\delta(\mathcal{P}) = \sum\limits_{k=1}^{n} \left[\sum\limits_{i=1}^{n(k)} \delta(\triangle_{k_i}) \right]$.

$\left[\sum\limits_{i=1}^{n(k)} \delta(\triangle_{k_i}) \right] = \delta(\mathcal{P}_k)$ by the result of ② and the earlier triangulation of $\mathcal{P}_k$.

$\therefore \delta(\mathcal{P}) = \sum\limits_{k=1}^{n} \delta(\mathcal{P}_k)$.

□