Mike Sorice **College Geometry II**
Spring 2019 **Homework 1**

① Let P be a point inside $\angle A = \angle \Sigma A \Omega$, with A an ordinary point and Σ and Ω ideal points. Let $\gamma = c(P, AP)$. Does γ always meet the sides of $\angle A$ at some ordinary points...

...in $\mathbb{E}^2$?

Proposition. *γ always meets both sides of $\angle A$.*

Proof. By Congruence Axiom 1, $\exists! S$ s.t. $S \in \overrightarrow{A\Sigma} \wedge AS \cong AP$.
As A and AP are respectively the center and radius of γ by construction, $AS \cong AP \Rightarrow S \in \gamma$ by definition of circle.
$\therefore \overrightarrow{A\Sigma}$ meets γ at S.
Similarly, $\exists! W \in \overrightarrow{A\Omega} \cap \gamma$.

□

Since all points of $\mathbb{E}^2$ are ordinary, so are S and W. Therefore, γ does meet both sides of $\angle A$ at ordinary points.

...in $\mathbb{H}^2$? We made no use of Euclid's Fifth Postulate in the above proof, so γ still meets both sides of $\angle A$ in $\mathbb{H}^2$. It may seem possible to construct counterexamples in models of $\mathbb{H}^2$ like this one:

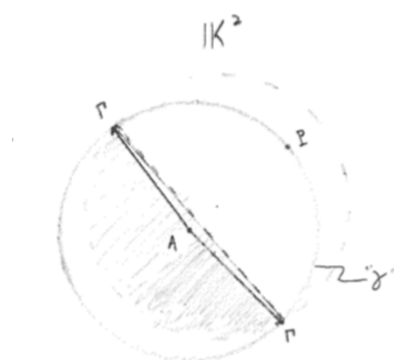

Figure 1: Seeming counterexample in $\mathbb{K}^2$. Γ and Λ are ideal points where "γ" meets the boundary. It seems that any angle whose sides are in the shaded area – on the side of $\Gamma\Lambda$ opposite P – will therefore miss "γ"...

However, these rely on flawed reasoning. In this case, a flaw is that , while"γ" is a circle in the larger Euclidean plane in which $\mathbb{K}^2$ is embedded, some points of "γ" are further from A in $\mathbb{K}^2$ than others. $AP < A\Lambda$, for instance[1]. Therefore, "γ" is by definition not a circle in $\mathbb{K}^2$.

[1] In fact $A\Lambda$ is infinite compared to AP!

Further, since all points finitely far from the normal point A are normal points, and since $AS \cong AW \cong AP$, S and W are normal point as long as P is. If, on the other hand, P is an ideal point, then so are S and W, since $\overline{AP}$ is then infinite, so that $\overline{AS}$ and $\overline{AW}$ are as well, but only ideal points are infinitely far from the normal point A.

In summary, then, in $\mathbb{K}^2$, $\gamma = c(A, AP)$ meets both sides of $\angle A$ at ordinary points as long as P is a normal point. On the other hand, γ meets neither side of $\angle A$ at ordinary points if P is an ideal point.

② Given $\angle A = \angle \Sigma A \Omega$ with A an ordinary point and Σ and Ω ideal points, find $\{P\}$ such that P is inside $\angle A$ and...

(i) ...$\exists l$ through P that meets neither side of $\angle A$...

... in $\mathbb{E}^2$.

Proposition. *$\{P\} = \emptyset$.*

Proof. $\overleftrightarrow{A\Sigma} = s$ and $\overleftrightarrow{A\Omega} = w$ are distinct and not parallel by construction (as a ray of each forms $\angle A$.)

There is a unique line through P parallel to $\overleftrightarrow{A\Sigma} = s$, say t. There is also a unique line through P parallel to $\overleftrightarrow{A\Omega} = w$, say u. Both results are by the Euclidean parallel postulate.

t is distinct from u as s is distinct from and not parallel to w and the relation "is parallel or equal to" is transitive for lines in $\mathbb{E}^2$ by previous result.

Any line through P is then exactly one of: t and parallel to s, u and parallel to w, or some third line parallel to neither s nor w.

Case 1 ($l = t$): l meets w as l cannot be parallel to both s and w since $s \neq w \wedge s \nparallel w$; $l \neq w$ since $P \notin w$, P being interior to $\angle A$, but $P \in l$ by hypothesis; and $(\parallel \vee =)$ is transitive by earlier result.

l does not meet s, so all points of l are on the same side of s, namely Ω's (and P's) side.

The ray of w on Ω's side of s is $\overrightarrow{A\Omega}$, so l meets $\overrightarrow{A\Omega}$, a side of $\angle A$. //

Case 2 ($l = u$): l meets s as l cannot be parallel to both s and w since $s \neq w \wedge s \nparallel w$; $l \neq s$ since $P \notin s$, P being interior to $\angle A$, but $P \in l$ by hypothesis; and $(\parallel \vee =)$ is transitive by earlier result.

l does not meet w, so all points of l are on the same side of w, namely Σ's (and P's) side.

The ray of s on Σ's side of w is $\overrightarrow{A\Sigma}$, so l meets $\overrightarrow{A\Sigma}$, a side of $\angle A$. //

Case 3 ($l \neq t \wedge l \neq u$): Then l meets both s and w.

If at A, then l meets both sides of $\angle A$. /

If not at A, then it meets s and w at two distinct points, say S and W. Exactly one of $S * P * W$, $P * S * W$, or $P * W * S$ by Betweenness Axiom 3 (P is neither S nor W as it lies on neither S nor W; $S \neq W$ as then $S = W = A$ which is false by hypothesis.)

If $S * P * W$, then, by plane separation and definition of ray, S is on P's side of w and therefore on $\overrightarrow{A\Sigma}$, a side of $\angle A$. Further, by the same reasoning, W is on P's side of s and therefore on $\overrightarrow{A\Omega}$, a side of $\angle A$. /
If $P * S * W$, then by the same reasoning, S is on P's side of w and therefore on $\overrightarrow{A\Sigma}$, a side of $\angle A$. /
Finally, if $P * W * S$, then by the same reasoning, W is on P's side of s and therefore on $\overrightarrow{A\Omega}$, a side of $\angle A$. //

Therefore, in all cases, any line through P meets at least one side of $\angle A$.

□

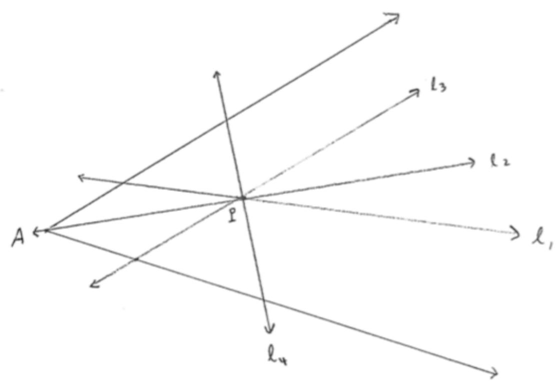

Figure 2: No Euclidean line through a point within $\angle A$ can miss its sides.

... in $\mathbb{H}^2$.

Proposition. *$\{P\}$ contains no point interior to $\triangle A\Sigma\Omega$.*

Proof. Suppose P is interior to $\triangle A\Sigma\Omega$.
Then any ray r from P meets a side of $\triangle A\Sigma\Omega$ by Proposition 3.9 (b) (any ray emanating from an interior point of a triangle meets a side of that triangle.)
Therefore, by Pasch's theorem, the line l of which r is half meets at least two sides of $\triangle A\Sigma\Omega$.
As all but one side of $\triangle A\Sigma\Omega$ is a side of $\angle A$, l meets at least one side of $\angle A$.

□

Proposition. *$\{P\}$ contains all points on $\overleftrightarrow{\Sigma\Omega}$*

Proof. $\Sigma\Omega$ is such a line. □

Proposition. *$\{P\}$ contains all points across $\overleftrightarrow{\Sigma\Omega}$ from A.*

Proof. There is at least one line through P parallel to $\overleftrightarrow{\Sigma\Omega}$, say l.
All points on l are on P's side of $\overleftrightarrow{\Sigma\Omega}$.

All points of $\overrightarrow{A\Sigma}$ and $\overrightarrow{A\Omega}$ are on A's side of $\overleftrightarrow{\Sigma\Omega}$.
Therefore, l meets neither $\overrightarrow{A\Sigma}$ nor $\overrightarrow{A\Omega}$, being the two sides of $\angle A$. $\square$
$\therefore \{P\}$ is the portion of $\angle A$'s interior not on A's side of $\overleftrightarrow{\Sigma\Omega}$, including $\overleftrightarrow{\Sigma\Omega}$ itself.

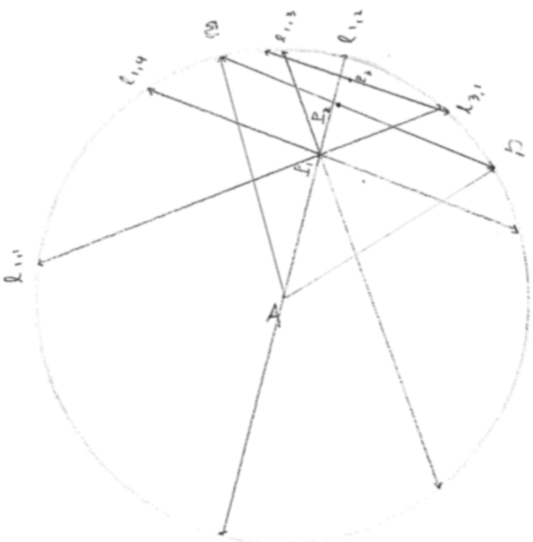

Figure 3: In the Beltrami-Klein model, no point within the $\frac{2}{3}$-ideal triangle with vertex A can have a line through it that misses the sides of $\angle A$, but points outside can.

(ii) ...$\exists l$ through P other than $\overleftrightarrow{AP}$ that meets both sides of $\angle A$...

... in $\mathbb{E}^2$.

Proposition. *$\{P\}$ is the whole interior of $\angle A$.*

Proof. Let P by some point within $\angle A$.
By the Euclidean Parallel Postulate, $\exists! m$ s.t. $(P \in m) \wedge [m \parallel (w = \overleftrightarrow{A\Omega})]$.
Since $m \nparallel (s = \overleftrightarrow{A\Sigma})$ (as $m \parallel w$ but $(w \nparallel s) \wedge (w \neq s)$) and $m \neq s$ (as $(A \in s) \wedge (A \notin m)$,) m meets s at some unique point by Proposition 2.1, say $S = m \cap s$.
Further, $S \in (\overrightarrow{A\Sigma} \backslash \{A\})$, since all points of m are on P's side of w, $\overrightarrow{A\Sigma}$ is the half of s on that side, and $A \in w$.
Let S' satisfy $A * S * S'$. Then $\overleftrightarrow{PS'}$ meets w since $\overleftrightarrow{PS'} \neq \overleftrightarrow{PS}$ (otherwise $S = S'$ since $\overleftrightarrow{A\Sigma} \cap m = S$, but $A * S * S' \Rightarrow S \neq S'$ by definition) and $\overleftrightarrow{PS}$ is the only line through P parallel to w. Say $W = w \cap \overleftrightarrow{PS'}$. Further let $l = \overleftrightarrow{S'W}$.
$\overrightarrow{AP}$ cuts $\angle \Sigma A \Omega$ (as P is within $\angle A$) so that $\angle \Sigma AP < \angle \Sigma A \Omega$ by definition. Let $\angle \Sigma A \Omega^\circ = \alpha$ and $\angle \Sigma AP^\circ = \alpha_1$ so that $\alpha > \alpha_1$.
Let $\angle APS = \phi$ and $\angle APS' = \phi'$. Now $\overrightarrow{PA} * \overrightarrow{PS} * \overrightarrow{PS'}$ (by $A * S * S'$ as $P \notin \overleftrightarrow{AS}$) so that $\angle APS' > \angle APS \Rightarrow \phi' > \phi$.

Let $\angle ASP^\circ = \sigma$ and $\angle AS'P^\circ = \sigma'$.
As Euclidean triangles have a common angle sum, $\Sigma(\triangle APS) = \alpha_1 + \phi + \sigma = \alpha_1 + \phi' + \sigma' = \Sigma(\triangle APS') \Rightarrow \sigma - \sigma' = \phi' - \phi > 0 \Rightarrow \sigma > \sigma' \Rightarrow \angle ASP > \angle AS'P$.
As s is a transversal of parallel lines m and w in $\mathbb{E}^2$, $\angle \Sigma A \Omega \cong \angle PSS'$ as alternate interior angles, so $\angle PSS'^\circ = \alpha$. Further, $\angle ASP$ supplements $\angle PSS'$, so $\alpha + \sigma = 180°$.
$\therefore \alpha + \sigma' < 180°$, so l meets w on P's side of s by Euclid's Fifth Postulate. Therefore, $W \in (\overrightarrow{A\Omega} \backslash \{A\})$. $\overrightarrow{A\Omega}$ being the other side of $\angle A$, l meets both sides of $\angle A$, intersects P, and does not intersect A.
As P is an arbitrary point within $\angle A$, what holds for P holds for any such point.
$\therefore$ all points within $\angle A$ have at least one line through other than $\overleftrightarrow{AP}$ that meets both sides of $\angle A$. $\square$

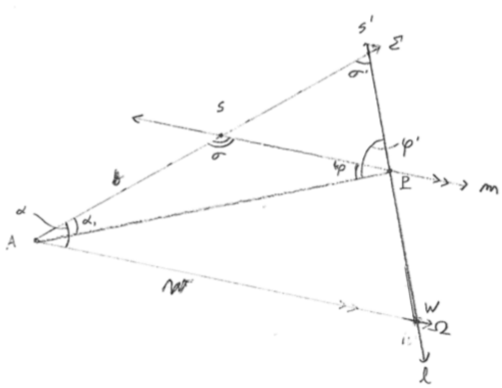

Figure 4: Euclidean parallelism makes it so that no point inside $\angle A$ can miss its sides.

... in $\mathbb{H}^2$.

Proposition. $\{P\}$ *includes all points within* $\triangle A\Sigma\Omega$..

Proof. Suppose P is within $\triangle A\Sigma\Omega$.
There is at least one line through P parallel to $\overleftrightarrow{\Sigma\Omega}$. Let l be such a line. Let r be a ray of l emanating from P. Then r meets a side of $\triangle A\Sigma\Omega$ by Proposition 3.9 (b). r does not meet $\Sigma\Omega$ as r is a ray of l and $l \parallel \overleftrightarrow{\Sigma\Omega}$ by construction. Further, r does not go through A as r would then be a ray of $\overleftrightarrow{AP}$, but $\overleftrightarrow{AP}$ must meet $\Sigma\Omega$ by the crossbar theorem and $l \parallel \overleftrightarrow{\Sigma\Omega}$ by construction. Therefore, r meets either $A\Sigma$ or $A\Omega$ at a point other than A, Σ, or Ω.
Then by Pasch's theorem, l meets another side of $\triangle A\Sigma\Omega$. By previous arguments, l does not meet $\Sigma\Omega$, nor does it intersect A. Therefore $-r$ meets whichever of $A\Sigma$ or $A\Omega$ that r does not meet.
Consequently l meets both sides of $\angle A$.

□

Proposition. $\{P\}$ *includes no point on* $\overleftrightarrow{\Sigma\Omega}$.

Proof. Suppose $P \in \overleftrightarrow{\Sigma\Omega}$.
Let $A * S * \Sigma$ and $l = \overleftrightarrow{PS}$.
Suppose for contradiction that $(W \in l) \wedge (W \in \overrightarrow{A\Omega})$.
Then $S * P * W$ by Proposition 3.7 (point on crossbar is interior if and only if it is between points on sides) as P is interior to $\angle A$ by hypothesis.
Then W is not on S's side of $\overleftrightarrow{\Sigma\Omega}$ by plane separation.
But all points of $\overrightarrow{A\Omega}$ are on S's side of $\overleftrightarrow{\Sigma\Omega}$ except for Ω itself, so l can only meet $\overrightarrow{A\Omega}$ at Ω.
However, since $P \in \overleftrightarrow{\Sigma\Omega} \wedge P \in l$, if l goes through Ω, then $l = \overleftrightarrow{\Sigma\Omega}$ so that $S = \Sigma$, contradicting $S * P * W$. $\Rightarrow\Leftarrow$
Similarly, if $A * W * \Omega$ and $l = \overleftrightarrow{PW}$, then l cannot meet $\overrightarrow{A\Sigma}$.
However, l must meet one of the sides of $\triangle A\Sigma\Omega$ by the crossbar theorem.
Therefore, no such l can meet both non-$\Sigma\Omega$ sides of $\triangle A\Sigma\Omega$, but these are the sides of $\angle A$. □

Proposition. $\{P\}$ *includes no point across* $\overleftrightarrow{\Sigma\Omega}$ *from* A.

Proof. Suppose P is on the side of $\overleftrightarrow{\Sigma\Omega}$ across from A.
Let $P \in l$. Then either l meets $\Sigma\Omega$ or not.
If l meets $\overleftrightarrow{\Sigma\Omega}$, then this is reduced to the previous case.
If on the other hand $l \parallel \overleftrightarrow{\Sigma\Omega}$, then l is entirely on P's side of $\overleftrightarrow{\Sigma\Omega}$ and can therefore meet neither side of $\angle A$ by plane separation, as the the entirety of both sides of $\angle A$ except Σ and Ω themselves is on A's side of $\overleftrightarrow{\Sigma\Omega}$.

□

$\therefore$ $\{P\}$ consists of all points interior to $\triangle A\Sigma\Omega$, that is, it is the complement in the interior of $\angle A$ of the set in part (i).

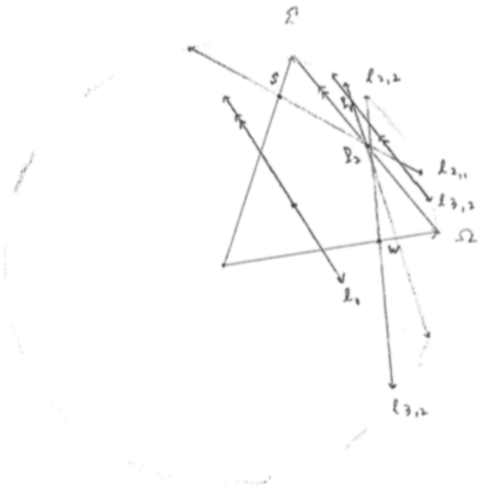

Figure 5: In $\mathbb{K}^2$, lines through P parallel to the limiting crossbar allow us to construct l that hits both sides if P is within the limiting crossbar, and prevent that construction if P is without.

(iii) ...$\forall l$ through P except $\overleftrightarrow{AP}, l$ meets exactly one side of $\angle A$...

... in $\mathbb{E}^2$.

Proposition. $\{P\} = \emptyset$.

Proof. As shown in (ii), every P interior to $\angle A$ has at least one line other than $\overleftrightarrow{AP}$ through that meets both sides of $\angle A$. □

... in $\mathbb{H}^2$.

Proposition. $\{P\} = \emptyset$.

Proof. As shown in (ii), every P interior to $\triangle A\Sigma\Omega$ has at least one line other than $\overleftrightarrow{AP}$ through that meets both sides of $\angle A$.
Further, as shown in (i), every P interior to $\angle A$ but not interior to $\triangle A\Sigma\Omega$ (i.e. on $\overleftrightarrow{\Sigma\Omega}$ or across $\overleftrightarrow{\Sigma\Omega}$ from A) has a line through that meets neither side. □

③ Given $\angle A$, is it possible $\forall R \in \mathbb{R}, R > 0$ to find an interior point B such that $c(B, R)$ is entirely within $\angle A$...

... in $\mathbb{E}^2$?

Proposition. *A circle of arbitrary radius can be made to fit within $\angle A$.*

Proof. Let $\angle A$ have sides $\overrightarrow{AB_1}$ and $\overrightarrow{AC_1}$. Suppose without loss of generality (by Congruence Axiom 1) that $AB_1 \cong AC_1$. Further let $b = \overline{AB_1} = \overline{AC_1}$ and $a = \overline{B_1C_1}$.
Every triangle has a unique incircle, a circle tangent to all sides, whose center is the point of intersection of its angle bisectors, say D_1. For $\triangle AB_1C_1$,

an isosceles triangle whose side lengths are $\{a, b, b\}$, the incircle has radius $r_1 = \frac{2A(\triangle)}{p(\triangle)} = \frac{2}{a+2b}\sqrt{\frac{a+2b}{2}\left(\frac{a+2b}{2}-a\right)\left(\frac{a+2b}{2}-b\right)\left(\frac{a+2b}{2}-b\right)} = \frac{ab}{a+2b}\sqrt{\left(\frac{a}{2b}\right)^2+1}$
$= \frac{a}{2}\frac{\sqrt{\left(\frac{a}{2b}\right)^2+1}}{\frac{a}{2b}+1}$
Any circle centered on D_1 whose radius is less than r_1 will fit entirely within $\triangle AB_1C_1$, and therefore within $\angle A$.

By Congruence Axiom 1, there is a point $B_n \in \overrightarrow{AB_1}$ such that $AB_n = n \cdot AB_1$ and likewise $C_n \in \overrightarrow{AC_1}$ such that $AC_n = n \cdot AC_1$. Thus $\overline{AB_n} = \overline{AC_n} = nb$.

$\triangle AB_nC_n$ being an isosceles triangle in $\mathbb{E}^2$ with $\angle A$ as one of its vertices, $\triangle AB_1C_1 \sim \triangle AB_nC_n$ so that $\overline{B_nC_n} = \frac{\overline{AB_n}}{\overline{AB_1}}\overline{B_1C_1} = n\overline{B_1C_1} = na$.

$\triangle AB_nC_n$ has an incircle centered at D_n, say, whose radius is $r_n = \frac{na}{2}\frac{\sqrt{\left(\frac{na}{2nb}\right)^2+1}}{\frac{na}{2nb}+1} =$
$n\frac{a}{2}\frac{\sqrt{\left(\frac{a}{2b}\right)^2+1}}{\frac{a}{2b}+1} = nr_1$.
Any circle centered at D_n whose radius is smaller than $r_n = nr_1$ will fit entirely within $\triangle AB_cC_n$.

As $r_1 > 0$, by the Archimedian property, for any real number R, there is some natural number n such that $nr_1 > R$. Therefore, $c(D_n, R)$ will fit entirely within $\triangle AB_nC_n$. The interior of $\triangle AB_nC_n$ being part of the interior of $\angle A$, *a fortiori* $c(D_n, R)$ fits entirely within $\angle A$.

□

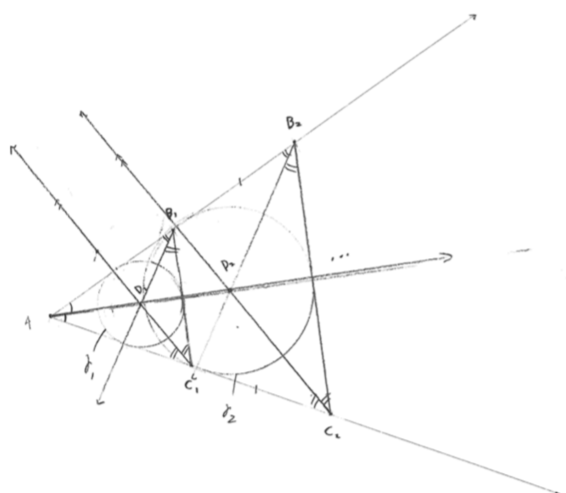

Figure 6: Crossbar B_iC_i may be moved down the sides of Euclidean angle $\angle A$ to fit an incircle of arbitrary size.

... in $\mathbb{H}^2$? We made use of Euclid V in the above proof for $\mathbb{E}^2$, but perhaps it was not necessary to do so. We must use a different route, then, to examine the case in $\mathbb{H}^2$.

Proposition. *There is a finite upper bound on the radius of any circle that can fit within* $\triangle A\Sigma\Omega$.

Proof. Let r bisect $\angle A$.

r meets $\Sigma\Omega$ by the crossbar theorem.

Further, $R = r \cap \Sigma\Omega$ is some regular point, since Σ and Ω are the only ideal points on $\Sigma\Omega$, but if r goes through either, it cannot bisect $\angle A$ (as then either $R = \Sigma$ so that $\angle RA\Omega \cong \angle\Sigma A\Omega \Rightarrow \angle RA\Omega > \angle RA\Sigma$, or $R = \Omega$ so that $\angle\Sigma AR \cong \angle\Sigma A\Omega \Rightarrow \angle RA\Sigma > \angle RA\Omega$, and either these contradict the construction of r as bisector of $\angle A$.)

Clearly no circle centered on r can fit inside $\triangle A\Sigma\Omega$ if its radius is bigger than AR, which is a finite segment as A and R are ordinary points.

Yet the incircle of $\triangle A\Sigma\Omega$ has its center on r and any circle larger than the incircle cannot be drawn entirely within $\triangle A\Sigma\Omega$.

$\therefore$ the largest circle that can be drawn in $\triangle A\Sigma\Omega$ is finite. $\square$

This does not entirely decide the matter as the half-plane across $\overleftrightarrow{\Sigma\Omega}$ from A is part of the interior of $\angle A$ yet not inside $\triangle A\Sigma\Omega$.

I speculate that a circle of arbitrary radius may be placed at some point on r across A from $\Sigma\Omega$.

Suppose that $c(B_1, r_1)$ fits within $\angle A$ such that $c(B_1, R_1 > r_1)$ is tangent to the sides of $\angle A$ (if tangent to one side at a point S_1 then it is also tangent to the other at a point W_1 as $\triangle AB_1S_1 \cong \triangle AB_1W_1$ by S.S.S. congruence.)

By Congruence Axiom 1, $\exists! B_2 \in r$ s.t. $AB_2 = 2 \cdot AB_1$ such that $B_2S_1 > B_1S_1$ and $c(B_2, R_1)$ now fits entirely within $\angle A$.

There will again be some R_2 so that $c(B_2, R_2)$ is tangent to the sides of $\angle A$, but this the process of finding a B_3 beyond B_2 may be repeated.

Further, the same process may be repeated an arbitrary number of times, so that there is a $B_n \in r$ s.t. $AB_n = n \cdot AB_1$ and $c(B_n, R_{n-1})$ fits in $\angle A$.

R_{n-1} grows without bound as B_n approaches the boundary of $\mathbb{K}^2$. Therefore, for any finite radius R, a point B_n can be found such that $R_{n-1} \leq R < R_n$, where R_n is defined such that $c(B_n, R_n)$ is tangent to the sides of $\angle A$. Then $c(B_n, R)$ fit entirely within $\angle A$, being itself entirely within $c(B_n, R_n)$.

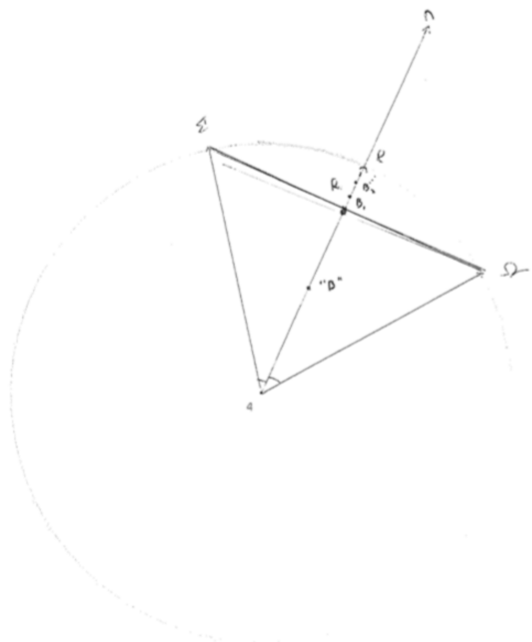

Figure 7: B_n can be placed arbitrarily close to the boundary of $\mathbb{K}^2$ to fit a circle of arbitrary finite radius.