SPRING 2019

MAT 3272 – COLLEGE GEOMETRY, Part II

Instructor: Dr. G. Galperin

FINAL EXAM

GEOMETRIES: Neutral $\mathbb{N}^2$, Euclidean $\mathbb{E}^2$, and Hyperbolic $\mathbb{H}^2$

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1. **(a)** Prove the transitivity of parallel lines in Euclidean geometry $\mathbb{E}^2$:
If $a||b$ and $b||c$, then $a||c$.

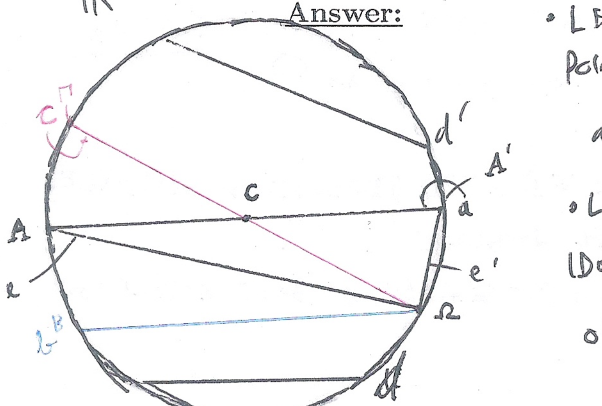
(TACITLY, $a \neq c$)

- SUPPOSE FOR CONTRADICTION $a \nparallel c$. THEN a & c MEET AT EXACTLY ONE POINT, SAY A.
- $A \in a$ & $a \| b$, SO $A \notin b$.
- BY EUCLID'S FIFTH POSTULATE, THERE IS A UNIQUE LINE THROUGH A PARALLEL TO b. ✱

$\therefore a \| c$. □

(b) In hyperbolic geometry $\mathbb{H}^2$, some three h-lines a, b, c satisfy the following three conditions: $a \asymp b$ (**divergently** parallel), $b \succ c$ (**asymptotically** parallel), <u>and</u> cXa ("X" means **intersecting** lines). Prove that there is a line d divergently parallel to each of the three lines a, b, c, and there is a line e asymptotically parallel to each of a, b, c. Use the Klein model $\mathbb{K}^2$ for your justification.

$\mathbb{K}^2$:



<u>Answer:</u>

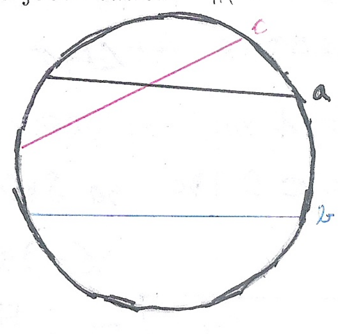
- LET $a = AA'$ AND $c \cap b = \{\Omega\}$, NECESSARILY AN IDEAL POINT. THEN $\underset{e''}{A\Omega}$ AND $\underset{e'}{A'\Omega}$ ARE BOTH $\succ$ TO ALL OF a, b, c. /
- LET $c = \Gamma\Omega$ AND $b = B\Omega$. THEN ANY LINE WHOSE IDEAL ENDPOINTS ARE ON $\overset{\frown}{\Gamma A'}$ OR $\overset{\frown}{\Omega B}$ IS $\asymp$ TO ALL OF a, b, c. // (SUCH AS d') (SUCH AS d)

□

(c) <u>True or False</u>: $\boxed{\text{In } \mathbb{H}^2, \text{ if } a \asymp b \text{ and } b \asymp c, \text{ then either } a \asymp c \text{ or } a \succ c.}$ If this statement is **True**, <u>justify it</u>; if it is **False**, give a <u>counterexample</u>. Circle your answer below. Use the Klein model $\mathbb{K}^2$ for your answer and justification. $\mathbb{K}^2$:

<u>Answer:</u> True **False**

IT CAN BE THAT $a X c$ EVEN THOUGH $a \asymp b$ & $b \asymp c$:



□.

2. Lines a and b have a common perpendicular PQ ($P \in a$, $Q \in b$). Point A is marked on the line a and point B on the line b such that the segment AB intersects the segment PQ at point N, where $BN < NA$. Denote $\angle PAN = \alpha$ and $\angle QBN = \beta$. Answer, **with a proof**, the following:

Which of the three possibilities can happen: $\alpha < \beta$, $\alpha = \beta$, or $\alpha > \beta$?

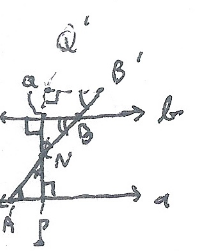
Answer: • ∡APN ≅ ∡BQN AS RIGHT ANGLES. ∡ANP ≅ ∡BNQ AS VERTICAL ANGLES.

• IN $\mathbb{E}^2$, THEN, △ANP ~ △BNQ BY A.A., SO (∡PAN ≅ ∡QBN) ⟺ ($\alpha=\beta$).

—

• IN $\mathbb{H}^2$, LAY A COPY OF AN OFF ON $\overrightarrow{NB}$ FROM N, PRODUCING B', THEN ERECT A PERPENDICULAR FROM $\overleftrightarrow{PQ}$ TO B', CALLING ITS FOOT Q'. THEN NB' ≅ NA BY CONSTRUCTION, ∡B'Q'N ≅ ∡APN AS RIGHT ANGLES, SO △ANP ≅ △B'NQ' BY A.A.S.

• ∴ ∡NAP ≅ ∡NB'Q.



• △BNQ ⊊ △B'NQ', SO $\delta(\triangle BNQ) < \delta(\triangle B'NQ')$

$$2e - (\cancel{e} + \cancel{x} + \beta) < 2e - (\cancel{e} + \cancel{x} + \alpha) \Rightarrow \boxed{\alpha < \beta}$$

∡ANP° = ∡BNQ°

∴ $\boxed{\text{EITHER } \alpha=\beta \ (\mathbb{E}^2) \text{ OR } \alpha<\beta \ (\mathbb{H}^2)}$. □

3. A circle **c** with center O is drawn in the hyperbolic plane $\mathbb{H}^2$, and three points A, B, C are marked on the circle **c** such that AB is a diameter. Answer the following two questions an **prove your answers**.

(a) Is the inscribed $\angle ACB = \angle C$ less, equal, or greater than a right angle? **Answer:** ∡C° < e

(b) Is the inscribed angle $\angle BAC = \angle A$ less, equal, or greater than $\widehat{BC}/2$?
Answer: ∡A° < $\widehat{BC}°/2$.

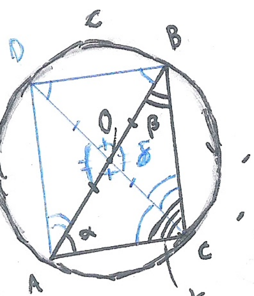
PROOF: • DRAW CD, A DIAMETER OF c. AO ≅ BO ≅ CO ≅ DO BY DEFINITION OF CIRCLE.

• ∡AOC ≅ ∡BOD & ∡AOD ≅ ∡BOC AS VERTICAL ANGLES.

• ∡ADO ≅ ∡DAO, ∡ACO ≅ ∡CAO, ∡BCO ≅ ∡CBO, & ∡BDO ≅ ∡DBO AS BASE ANGLES OF ISOSCELES TRIANGLES.

• △AOC ≅ △BOD & △AOD ≅ △BOC BY S.A.S.

• LET α = ∡BAC°, β = ∡ABC°, & γ = ∡ACB°. THEN, $\gamma = \alpha+\beta$ AND Σ(□ACBD) = 4γ. BUT Σ(□ACBD) < 4e BY COROLLARY OF SACCHERI-LEGENDRE THEOREM SO $4\gamma < 4e$ ⟹ $\gamma < e$ ⟹ $\boxed{\text{∡ACB IS ACUTE}}$.



• $\widehat{BC}°$ = ∡BOC° BY DEFINITION.

• △BOC ⊂ △ABC, SO $\delta(\triangle BOC) < \delta(\triangle ABC)$

$$\cancel{2e} - (2\cancel{\beta} + \delta) < \cancel{2e} - (\alpha + \cancel{\beta} + \alpha + \cancel{\beta}) \Rightarrow \alpha < \delta/2 = \widehat{BC}°/2.$$

∡BOC°

∴ $\boxed{\text{∡BAC}° < \widehat{BC}°/2}$. □

4. Implement the following instruction and answer, with a proof, to the questions (a) and (b) below.

   (0) Draw the Klein model $\mathbb{K}^2$ of hyperbolic plane $\mathbb{H}^2$ and draw the horizontal and vertical diameters through the center O of the model (consider them as the $x-$ and the $y-$ axes).

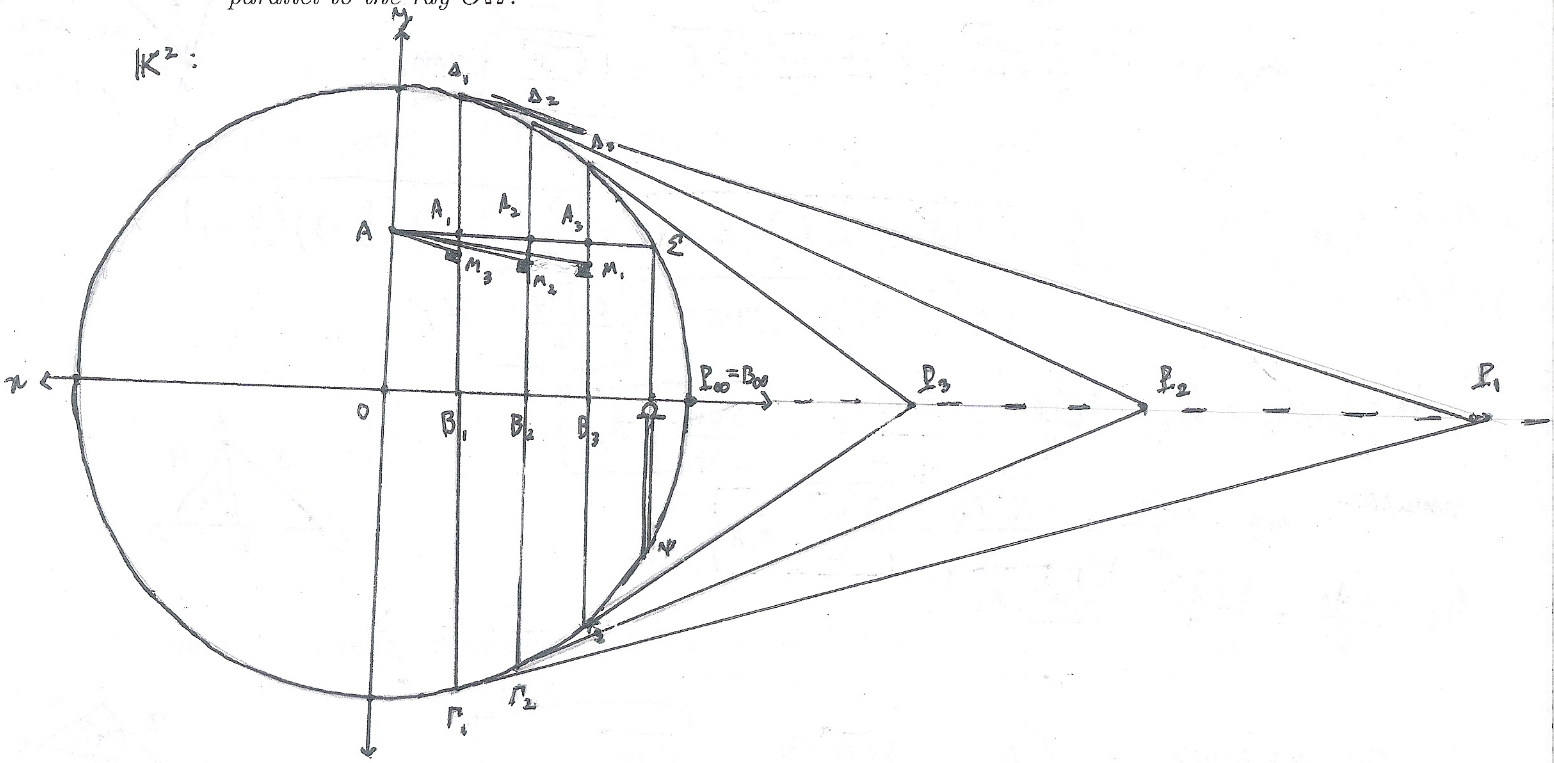
   (1) In the first quadrant of the circle, mark the midpoint A on the upper vertical radius (where $y > 0$) and then draw the horizontal half-chord $A\Sigma$ to the right through point A.

   (2) Draw some three vertical half-chords in the first quadrant and label the points of their intersection with the half-chord $A\Sigma$ as A_1, A_2, and A_3; draw also the vertical chord down through point Σ. Label by B_1, B_2, and B_3 the intersection points of the three drawn vertical half-chords with the positive $x-$axis, and by Ω the half-chord through Σ (so the point Ω lies on the $x-$axis).

   (3) Consider now the Euclidean segments A_1B_1, A_2B_2, A_3B_3, and $\Sigma\Omega$ as hyperbolic segments of the respective hyperbolic lines in the model $\mathbb{K}^2$. Construct the **hyperbolic** perpendiculars AM_1, AM_2, and AM_3 from point A to the **hyperbolic** lines A_1B_1, A_2B_2, A_3B_3, respectively.

   (a) *Prove that the lines your draw are indeed the h-perpendiculars in your construction.*

   (b) *Suppose you drew infinitely many such h-perps A_nB_n, $n = 1, 2, 3, \ldots$ (as you did for the first three of them). Do the hyperbolic rays $\overrightarrow{AM_n}$ tend asymptotically to the hyperbolic ray $\overrightarrow{O\Omega}$ as $n \to \infty$? Or the hyperbolic ray $\overrightarrow{AM} = \lim_{n\to\infty} \overrightarrow{AM_n}$ is divergently parallel to the ray $\overrightarrow{O\Omega}$?*



a) AM_i IS AN $\mathbb{E}^2$ SEGMENT OF AP_i, WHERE P_i IS THE POLE OF $\Delta_i\Gamma_i$, THE $\mathbb{H}^2$-LINE OF WHICH A_iB_i IS A SEGMENT.

$\overleftrightarrow{AP_i} \perp_{\mathbb{H}^2} \overleftrightarrow{A_iB_i}$ BY CONSTRUCTION. IN $\mathbb{K}^2$.

b) $\boxed{\lim_{n\to\infty} \overrightarrow{AM_n} \geq \overrightarrow{O\Omega}.}$ $P_n \xrightarrow{n\to\infty} P_\infty$, WHICH IS THE IDEAL POINT OF $\overrightarrow{O\Omega}$, AS LONG AS $B_n \xrightarrow{n\to\infty} (B_\infty = P_\infty)$ AS WELL.

5. Let $\triangle ABC$ be an **obtuse** Euclidean triangle with the side lengths $BC = a$, $AC = a+1$, and $AB = a+2$.

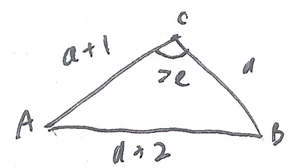
(a) Find the range for the length of each side. **Answer:** $\boxed{\begin{array}{c} 1<a<3 \\ 2<a+1<4 \\ 3<a+2<5 \end{array}}$

FOR AN OBTUSE Δ:

$$(a+2)^2 > (a+1)^2+a^2 = 2a^2+2a+1 \iff a^2-2a-3=(a+1)(a-3)<0 \Rightarrow -1<a<3.$$

BY THE Δ INEQUALITY:

$$a+a+1>a+2 \iff 2a+1>a+2 \iff a>1.$$



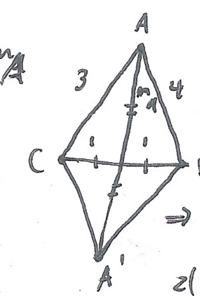
(b) If all the three side lengths are **integers**, find the lengths of the three medians m_A, m_B, and m_C. THEN $a=2$, $\underbrace{a+1}_{b}=3$, $\underbrace{a+2}_{c}=4$.

Answer:

THEN $m_A = \dfrac{\sqrt{2b^2+2c^2-a^2}}{2} = \dfrac{\sqrt{2\cdot 9+2\cdot 16-2\cdot 2}}{2} = \boxed{\dfrac{\sqrt{46}}{2}} = m_A$

$m_B = \dfrac{\sqrt{2a^2+2c^2-b^2}}{2} = \dfrac{\sqrt{2\cdot 4+2\cdot 16-9}}{2} = \boxed{\dfrac{\sqrt{31}}{2}} = m_B$

$m_C = \dfrac{\sqrt{2a^2+2b^2-c^2}}{2} = \dfrac{\sqrt{2\cdot 4+2\cdot 9-2\cdot 8}}{2} = \boxed{\dfrac{\sqrt{10}}{2}} = m_C.$



$\Rightarrow (2m_A)^2+2^2 = 2(3^2+4^2).$

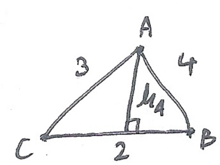
(c) In the frame of item (b), find the area of $\triangle ABC$ and the lengths of its three altitudes h_A, h_B, and h_C.

$a+b+c=2s$, so $s=9/2$.

Answer: $A_\Delta = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{\frac{9}{2}\left(\frac{9}{2}-2\right)\left(\frac{9}{2}-3\right)\left(\frac{9}{2}-4\right)}$

$= \dfrac{3}{4}\sqrt{(9-4)(9-6)(9-8)} = \boxed{\dfrac{3\sqrt{15}}{4}} = A_\Delta.$

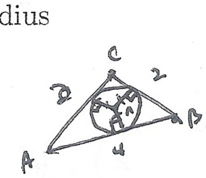
$A_\Delta = \frac{1}{2}a h_A \Rightarrow h_A = \dfrac{2A_\Delta}{a} = \dfrac{3\sqrt{15}/2}{2} = \boxed{\dfrac{3\sqrt{15}}{4} = h_A}$



SIMILARLY, $h_B = \dfrac{2A_\Delta}{b} = \dfrac{3\sqrt{15}/2}{3} = \boxed{\dfrac{\sqrt{15}}{2} = h_B}$

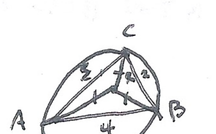
$h_C = \dfrac{2A_\Delta}{c} = \dfrac{3\sqrt{15}/2}{4} = \boxed{\dfrac{3\sqrt{15}}{8} = h_C}.$

(d) In the frame of item (b), find the radius of the circumscribed circle and the radius of the inscribed circle for $\triangle ABC$. **Answer:**



THE INRADIUS IS $r = \dfrac{A_\Delta}{s} = \dfrac{3\sqrt{15}/4}{9/2} = \boxed{\dfrac{\sqrt{15}}{6} = r}.$

THE CIRCUMRADIUS IS $R = \dfrac{abc}{4A_\Delta} = \dfrac{2\cdot 3\cdot 4}{4\cdot 3\sqrt{15}/4} = \boxed{\dfrac{8\sqrt{15}}{15} = R.}$



6. Prove for the Neutral geometry $\mathbb{N}^2$ that a line ℓ cannot intersect a circle $\mathbf{c}$ at 3 or more than 3 distinct points.

(Only after proving this fact, saying that <u>at most</u> 2 intersection points can occur with a line and a circle, one can give a correct definition of the interior of the circle $\mathbf{c}$. So, the use of the undefined term "interior" is forbidden in your proof.)

LET ℓ MEET c AT DISTINCT POINTS A & B, & c HAVE CENTER O, SO THAT $OA \cong OB$.

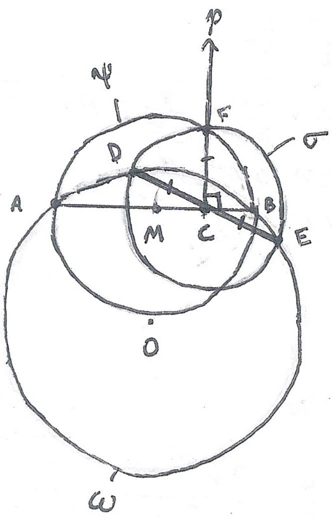
- CASE 1: IF $O \in \ell$, THEN $A * O * B$ & IF $OC \cong OA$, $C = B$ & VICE VERSA BY AXIOM.
- CASE 2: $O \notin \ell$. THEN DROP A PERPENDICULAR TO O FROM ℓ, CALLING ITS FOOT F. $\angle AFO \cong \angle BFO$ AS RIGHT ANGLES, SO $\triangle AFO \cong \triangle BFO$ BY H.-L. (FO IS A COMMON LEG)

$\therefore$ $AF \cong BF$ WITH $A * F * B$.

• SUPPOSE $C \in c \cap \ell$. THEN $\triangle CFO \cong \triangle AFO \cong \triangle BFO$ ($\because CO \cong AO \cong BO$), SO $C = A$ OR $C = B$, SINCE $A * F * B$ & $AF \cong BF \cong CF$. $\square$

7. Let AB be a horizontal segment in Euclidean plane. Point C is marked inside the segment AB so that $AC = a$ and $CB = b$. A circle ω of an unknown radius is drawn through the ends A and B of the segment AB; its center is a known point O. You can draw an arbitrary circle σ centered at point C and denote by D and E the intersection points where the circle σ meets the circle ω.

(a) Draw by compass and straightedge the circle σ centered at point C such that its intersection points D and E with the circle ω are the ends of **a diameter** of the circle σ. Enumerate and describe the steps of your construction.



1) ERECT A PERPENDICULAR TO AB AT C, CALL IT p.

2) DRAW CIRCLE ψ WITH AB AS ITS DIAMETER. ($\psi = O(M, AM)$ WITH $A * M * B$, $AM \cong BM$.)

3) $\{F\} = p \cap \psi$.

4) $\sigma = O(C, CF)$. $\{D, E\} = \sigma \cap \omega$.

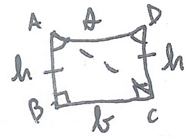
$\overline{CF} = \sqrt{ab}$, WHICH IS THE SQUARE ROOT OF THE POWER OF C IN ω, SO $CF \cong CD \cong CE$ WITH $\{D, E\} = \sigma \cap \omega$.

(b) Determine the radius r of the circle σ in terms of the lengths a and b.

Answer: $r = \sqrt{ab}$

BY POWER OF A POINT THEOREM:

$P_\omega(C) = a \cdot b = \overline{CD} \cdot \overline{CE} = r^2$.



8. (a) Let $ABCD$ be a Saccheri quadrilateral with the base BC. **Prove:**

$$AD - BC < 2 \cdot AB.$$

LET $\overline{AB} = \overline{CD} = h$, $\overline{BC} = b$, $\overline{AD} = a$.

• DRAW THE DIAGONAL AC & CONSIDER $\triangle ABC$ & $\triangle ACD$.

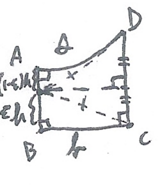
– THE $\triangle$ INEQUALITY ON $\triangle ABC$ GIVES: $b + h > \overline{AC}$

– THE $\triangle$ " " $\triangle ACD$ " : $\overline{AC} + h > a$

$$+\quad b + \cancel{\overline{AC}} + 2h > a + \cancel{\overline{AC}} \Rightarrow 2h > a - b$$

$\therefore$ $\boxed{\overline{AD} - \overline{BC} < 2 \cdot \overline{AB}}$ $\square$

(b) Let $ABCD$ be a ~~Lambert~~ quadrilateral with the acute angle at the vertex D. Prove: $AD - BC < AB$.



FOR NOTATIONAL CONVENIENCE, LET $h = \overline{AB}$, $b = \overline{BC}$, $a = \overline{AD}$.

• CHOOSE $E^{*} \in AB$ SUCH THAT $CE \cong DE$.

SAY THIS PARTITIONS AB SO THAT $\overline{BE} = \varepsilon h$ & $\overline{AE} = (1-\varepsilon)h$ FOR SOME $\varepsilon \in (0,1)$.

• CONSIDER $\triangle BCE$ & $\triangle ADE$.

– THE $\triangle$ INEQ. ON $\triangle BCE$ GIVES: $b + \varepsilon h > \overline{CE}$

– " " " " $\triangle ADE$ " : $\overline{DE} + (1-\varepsilon)h > a$

$$+\quad b + h + \overline{DE} > \overline{CE} + a \Rightarrow h > a - b + \cancel{\overline{CE} - \overline{DE}}$$

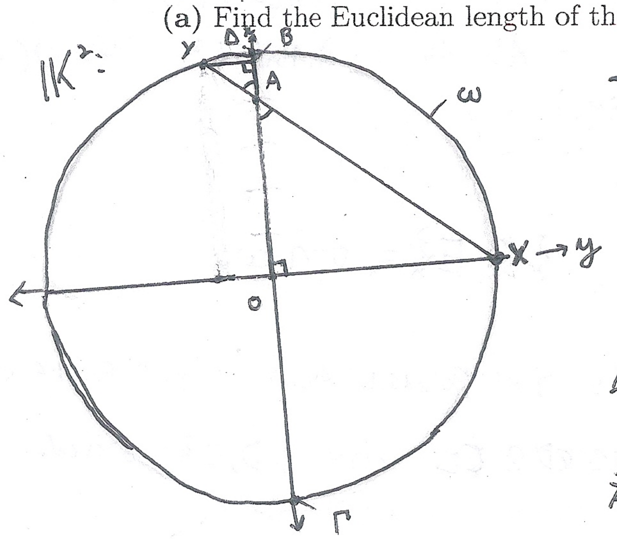
$\overline{CE} - \overline{DE} \to 0$ AS $CE \cong DE$ BY CONSTRUCTION.

$\therefore$ $\boxed{\overline{AD} - \overline{BC} < \overline{AB}}$ $\square$

$^{*}\{E\} = CD^{\perp} \cap AB$

9. Consider the unit circle ω centered at the origin of the $xy-$plane as the Klein model $\mathbb{K}^2$ of hyperbolic plane $\mathbb{H}^2$. Let $A = \left(\frac{4}{5}, 0\right)$ and $X = (0,1)$. Draw the chord XY passing through point A and drop the perpendicular YB from Y to the $x-$axis.

(a) Find the Euclidean length of the segment AB. **Answer:**



– THE POWER OF A IS: $P_\omega(A) = \underbrace{\frac{1}{5}}_{\overline{A\Delta}} \cdot \underbrace{\frac{9}{5}}_{\overline{A\Gamma}} = \frac{9}{25} = \overline{AX} \cdot \overline{AY}$.

– BY THE PYTHAGOREAN THEOREM ON $\triangle AOX$,

$$\overline{AX} = \sqrt{\left(\frac{4}{5}\right)^2 + 1^2} = \sqrt{41}/5, \text{ so}$$

$$\overline{AY} = \frac{9}{5\sqrt{41}} = \frac{9\sqrt{41}}{205}.$$

$\triangle AOX \sim \triangle ABY$ (A.A.~), so

$$\frac{\overline{AB}}{\overline{AY}} = \frac{\overline{AO}}{\overline{AX}} \Rightarrow \overline{AB} = \frac{\overline{AY}}{\overline{AX}}\overline{AO} = \frac{4}{5} \cdot \frac{9\sqrt{41}/205}{\sqrt{41}/5} = \boxed{\frac{36}{205} = \overline{AB}}.$$

(b) Find the hyperbolic lengths $||OA||$ and $||AB||$ of the segments OA and AB. Which length is bigger, $||OA||$ or $||AB||$? **Answer:**

$$||AB|| = \frac{1}{2}\left|\ln\left(\frac{\overline{\Gamma B} \cdot \overline{A\Delta}}{\overline{\Gamma A} \cdot \overline{B\Delta}}\right)\right|.$$

$$\overline{B\Delta} = \underbrace{\frac{1}{5}}_{\overline{A\Delta}} - \underbrace{\frac{36}{205}}_{\overline{AB}} = \frac{41 - 36}{205} = \frac{5}{205} = \frac{1}{41}. \quad \overline{\Gamma B} = \underbrace{\frac{9}{5}}_{\overline{\Gamma A}} + \underbrace{\frac{36}{205}}_{\overline{AB}} = \frac{9 \cdot 41 + 36}{205} = \frac{405}{205} = \frac{81}{41}.$$

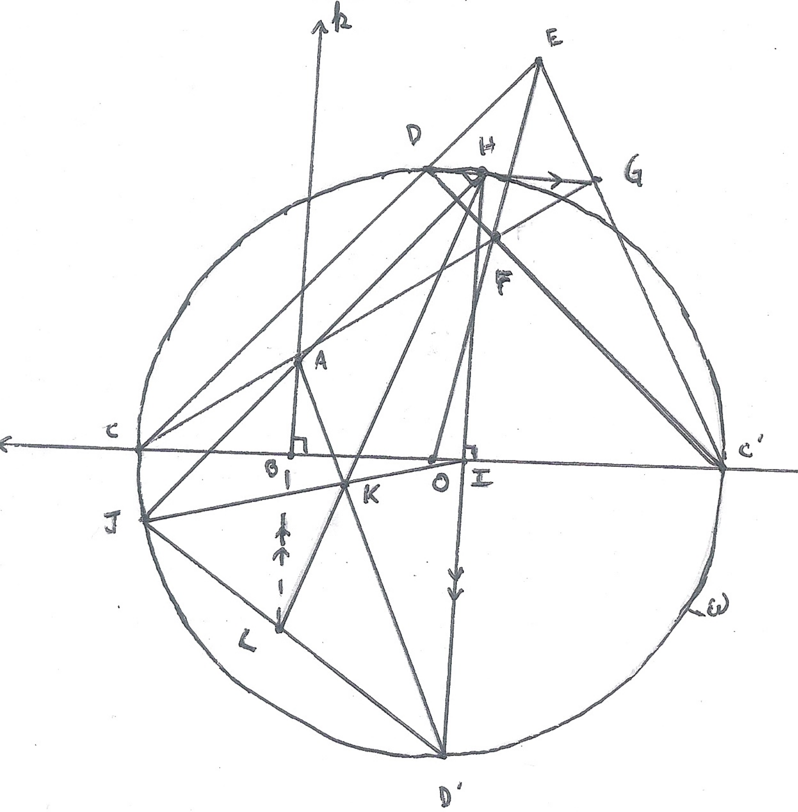
$$\therefore ||AB|| = \frac{1}{2}\left|\ln\left(\frac{81/41 \cdot 1/5}{9/5 \cdot 1/41}\right)\right| = \frac{1}{2}\ln 9 = \boxed{\ln 3} = ||AB||.$$

$$||OA|| = \frac{1}{2}\left|\ln\left(\frac{\overline{\Gamma A} \cdot \overline{O\Delta}}{\overline{\Gamma O} \cdot \overline{A\Delta}}\right)\right| = \frac{1}{2}\left|\ln\left(\frac{9/5 \cdot 1}{1 \cdot 1/5}\right)\right| = \frac{1}{2}\ln 9 = \boxed{\ln 3 = ||OA||}.$$

THEY'RE EQUAL!

$AB \cong_{\mathbb{H}^2} OA$.

10. Let ℓ be a horizontal line in the Euclidean plane and ω be a circle of some radius centered at point $O \in \ell$. Point A is located inside the circle ω in the upper half plane with the border ℓ. PROBLEM: Construct the perpendicular line $k = \overleftrightarrow{AB}$ from point A to the line ℓ ($B \in \ell$) using ONLY **straightedge** (and DO NOT use compass in your construction). Justify your construction, i.e. prove that the line $\overleftrightarrow{AB}$ is perpendicular to the line ℓ, indeed.



1) MARK $\{C, C'\}$ AS $\ell \cap \omega$. $CO \cong C'O$ AS $\omega \wedge O$.

2) CHOOSE $D \in \omega / \ell$. MARK D' OPPOSITE D THROUGH O.

3) CHOOSE E SUCH THAT $C * D * E$. DRAW CE, OE, $C'E$, & $C'D$.

4) MARK $\{F\} = C'D \cap OE$. DRAW $\overrightarrow{CF}$

5) MARK $\{G\} = \overrightarrow{CF} \cap C'E$. $\overleftrightarrow{DG} \parallel \ell$.*

6) MARK $\overrightarrow{DG} \cap \omega = \{H\}$. DRAW HD'. $HD' \perp \ell$.† IF $A \in HD'$, FINISHED. ELSE:

7) MARK $HD' \cap \ell = \{I\}$. $HI \cong D'I$ AS $\triangle OIH \cong \triangle OID'$ BY H.-L. ($OH \cong OD'$ AS $\omega \wedge O$, OI IS COMMON, $\measuredangle OIH$ & $\measuredangle OID'$ ARE ∟s.)

8) DRAW $\overrightarrow{HA}$. MARK $\overrightarrow{HA} \cap \omega = \{J\}$

9) DRAW JI, JD', AD'. MARK $AD' \cap JI = \{K\}$.

10) DRAW $\overrightarrow{HK}$. MARK $\overrightarrow{HK} \cap JD' = \{L\}$

11) $\overleftrightarrow{LA} \parallel \overleftrightarrow{HD'}$,‡ SO $\overleftrightarrow{LA} \perp \ell$ AS WELL.

MARK $\overrightarrow{LA} \cap \ell = \{B\}$. THEN $\overrightarrow{BA} = k$. □

* CONSIDER $\triangle CEC'$ & $\triangle DEG$. BY CEVA'S THEOREM ON $\triangle CEC'$, $\cancel{\frac{\overline{CI}}{\overline{C'I}}}^{1} \cdot \frac{\overline{C'G}}{\overline{EG}} \cdot \frac{\overline{DE}}{\overline{CD}} = 1$, SO

$$\frac{\overline{C'G}}{\overline{EG}} = \frac{\overline{CD}}{\overline{DE}} \Leftrightarrow \frac{\overline{C'G}}{\overline{EG}} + 1 = \frac{\overline{C'G} + \overline{EG}}{\overline{EG}} = \frac{\overline{C'E}}{\overline{EG}} = \frac{\overline{CD}}{\overline{DE}} + 1 = \frac{\overline{CD} + \overline{DE}}{\overline{DE}} = \frac{\overline{CE}}{\overline{DE}}$$ SO, AS $\measuredangle CEC'$ IS COMMON,

$\triangle CEC' \sim \triangle DEG$ (S.A.S. ~) & $\overleftrightarrow{DG} \parallel \overleftrightarrow{CC'}$ AS $\measuredangle EDG \cong \measuredangle ECC'$ (~~ALT.~~ INT. ∡S THM.)

† DD' IS A DIAMETER BY CONSTRUCTION, SO $\measuredangle DHD'$ IS A ∟ BY THALES' THM. BUT $\overleftrightarrow{DH} \parallel \ell$, SO $\measuredangle DHD' \cong \measuredangle CID'$ BY CONV. ALT. INT. ∡ THM.

‡ BY SIMILAR REASONING AS *.