MAT 3271: Propositions 2.1 & 2.2 Mike Sorice

Proposition (2.1)**.** *Given* $m, l \ni m \nparallel l$ *&* $m \neq l$*, then* $\exists! P \ni P \text{ I } m$ *&* $P \text{ I } l$*.*

Proof. Suppose $\nexists P \ni P \text{ I } m$ & $P \text{ I } l$. Then, $m \parallel l$ by definition. $l \nparallel m$ by hypothesis. $\Rightarrow\!\Leftarrow$. $\therefore \exists P \ni P \text{ I } m$ & $P \text{ I } l$.

Suppose $\exists Q \neq P \ni Q \text{ I } m$ & $Q \text{ I } l$. Then $P \text{ I } m$ & $Q \text{ I } m$ & $P \text{ I } l$ & $Q \text{ I } l$. $P \neq Q$ by supposition. $\therefore$, $\exists! n \ni P \text{ I } n$ & $Q \text{ I } n$ by Incidence Axiom 1. $\therefore l, m = n$ by definition. $\therefore l = m$ by the transitive property of $=$. However $l \neq m$ by hypothesis. $\Rightarrow\!\Leftarrow$. $\therefore \nexists Q \neq P \ni Q \text{ I } m$ & $Q \text{ I } l$. $\therefore P$ is unique by definition. □

Lemma. *There exist at least three distinct lines.*

Proof. $\exists$ distinct $P, Q, R \ni \nexists o \ni P \text{ I } o$ & $Q \text{ I } o$ & $R \text{ I } o$ by Incidence Axiom 3. $\therefore \exists! l, m, n \ni P \text{ I } l$ & $Q \text{ I } l$ & $P \text{ I } m$ & $R \text{ I } m$ & $Q \text{ I } n$ & $R \text{ I } n$ & $R \not\text{I}\, l$ & $Q \not\text{I}\, m$ & $P \not\text{I}\, n$ by Incidence Axiom 1. □

Proposition (2.2)**.** $\exists$ *distinct* l, m, n *that are not concurrent.*

Given $l \neq m$, either $l \parallel m$ or $l \nparallel m$ by the law of the excluded middle.

Proof of case $l \parallel m$. $l \parallel m \Rightarrow \nexists P \ni l \text{ I } P$ & $m \text{ I } P$ by definition. $\therefore \forall n \nexists P \ni P \text{ I } l$ & $P \text{ I } m$ & $P \text{ I } n$ a fortiori. $\exists n \neq l, m$ by lemma. $\therefore l, m$, and n are not concurrent by definition. $\therefore \exists l, m, n$ distinct and not concurrent. □

Proof of case $l \nparallel m$. $l \nparallel m \Rightarrow \exists! P \ni P \text{ I } l$ & $P \text{ I } m$ by Proposition 2.1. $\exists Q \neq P \ni Q \text{ I } l$ & $\exists R \neq P \ni R \text{ I } m$ by Incidence Axiom 2. Further, $Q \not\text{I}\, m$ & $R \not\text{I}\, l$ as P is unique. Further still, $Q \neq R$ as otherwise $Q \text{ I } m$ since $R \text{ I } m$.

Now, $\exists! n \ni Q \text{ I } n$ & $R \text{ I } n$ by Incidence Axiom 1. $n \neq l$ as $R \text{ I } n$ by definition, but $R \not\text{I}\, l$. $n \neq m$ as $Q \text{ I } n$ by definition, but $Q \not\text{I}\, m$. $\therefore l, m, n$ are distinct.

Suppose $P \text{ I } n$. Then, $P \text{ I } n$ & $Q \text{ I } n$. But $P \text{ I } l$ & $Q \text{ I } l$ and l is unique by Incidence Axiom 1, so $l = n$. $l \neq n$. $\Rightarrow\!\Leftarrow$. $\therefore P \not\text{I}\, n$. P is unique, so $\forall Q$, either $Q = P \Rightarrow Q \text{ I } l, m$ & $Q \not\text{I}\, n$ or $Q \neq P \Rightarrow Q \not\text{I}\, l$ or $Q \not\text{I}\, m$. $\therefore \nexists Q \ni Q \text{ I } l, m, n$. $\therefore \exists l, m, n$ distinct and not concurrent. □