MAT 2800: Assignment 1 Mike Sorice

1.2 Exercises

1. Given a convex n-gon p whose vertices are $A_1, A_2, ..., A_n$.

   Definition 1 (Adjacent vertices)**.** Two vertices of a polygon are *adjacent* if and only if they are both endpoints of a single side.

   Lemma. *Each vertex is adjacent to exactly 2 other vertices.*

   Proof. Each vertex is the end point of exactly 2 sides of the polygon by definition. □

   Stipulate that A_i is adjacent to $A_{(i-1) \bmod n}$ and $A_{(i+1) \bmod n}$. Thus, A_1 is adjacent to A_n and A_2, A_2 is adjacent to A_3 and A_1, etc. This stipulation is made without loss of generality as, if it does not hold for some indexed set of vertices $\{A_i\}$, we can re-label the set $\{A_{i'}\}$ by selecting any vertex as $A_{1'}$, labeling its adjacent vertices as $A_{2'}$ and $A_{n'}$, etc.

   Choose some point, say A_1, and draw the diagonals from A_1, i.e. line segments from A_1 to all non-adjacent vertices. This will produce the following set of triangles:

$$\begin{aligned} T &= \{\triangle A_1A_2A_3, \triangle A_1A_3A_4, \triangle A_1A_4A_5, ..., \triangle A_1A_{n-1}A_n\} \\ &= \bigcup_{i=2}^{n-1} \{\triangle A_1A_iA_{i+1}\}. \end{aligned} \tag{1.1}$$

   T clearly contains $n-2$ triangles. Further, each side of p appears exactly once in T as the two outer triangles ($\triangle A_1A_2A_3, \triangle A_1A_{n-1}A_n$) have sides consisting of 2 sides of p and one diagonal, whereas each of the $n-4$ inner triangles have sides consisting of 1 side of p (that between the higher-index vertices) and 2 diagonals (those between the higher-index vertices and A_1.) Therefore, the non-diagonal sides of T are exactly the same as the sides of p.

   Now consider the angles of the triangles in T. Let $\angle A_i^{(p)}$ denote the (interior) angle at vertex A_i in p and let $\angle A_i^{(j)}$ denote the angle at vertex A_i in the j^{th} triangle from T. Now, A_2 and A_n are vertices only of the outer triangles and therefore $\angle A_2^{(p)} = \angle A_2^{(1)}$ and $\angle A_n^{(p)} = \angle A_n^{(n-2)}$ as the angles are formed of the very same sides. A part of $\angle A_1^{(p)}$ appears in each triangle in T as it is cut by the diagonals into the parts $\left\{\angle A_1^{(1)}, \angle A_1^{(2)}, ..., \angle A_1^{(n)}\right\}$. This is true as p is convex, so $\angle A_1^{(p)}$ is always intersected by the ray emanating from A_1 which is an extension of a given diagonal. Similarly, each of the other angles is cut into two parts by the diagonal from A_1 to its corresponding vertex, such that $\angle A_i^{(p)}$ for $i \in \mathbb{N}$ and $2 < i < n$ is made of the two parts $\angle A_i^{(i-2)}$ and $\angle A_i^{(i-1)}$. Again, this is true as p is convex, so the diagonal from A_1 to A_i intersects $\angle A_i^{(p)}$, dividing it into the two parts listed.

By the triangle angle sum theorem (valid in the Euclidean plane,) the measures of the angles in each triangle in T sum to $180°$. As there are $n-2$ such triangles as shown above, the sum of all such angle measures is $180(n-2)°$. Therefore, we have:

$$\begin{aligned}
180(n-2)° =& m\angle A_1^{(1)} + m\angle A_2^{(1)} + m\angle A_3^{(1)} \\
&+ \sum_{i=2}^{n-3} \left(m\angle A_1^{(i)} + m\angle A_{i+1}^{(i)} + m\angle A_{i+2}^{(i)} \right) \\
&+ m\angle A_1^{(n-2)} + m\angle A_{n-1}^{(n-2)} + m\angle A_n^{(n-2)}. \\
=& \underbrace{\sum_{i=1}^{n-2} m\angle A_1^{(i)}}_{m\angle A_1^{(p)}} + m\angle A_2^{(p)} + \underbrace{m\angle A_3^{(1)} + m\angle A_3^{(2)}}_{m\angle A_3^{(p)}} \\
&+ \underbrace{m\angle A_4^{(2)} + m\angle A_4^{(3)}}_{m\angle A_4^{(p)}} + ... + \underbrace{m\angle A_{n-1}^{(n-3)} + m\angle A_{n-1}^{(n-2)}}_{m\angle A_{n-1}^{(p)}} \\
&+ m\angle A_n^{(p)} \\
=& \sum_{i=1}^{n} m\angle A_i^{(p)},
\end{aligned} \tag{1.2}$$

but $\sum_{i=1}^{n} m\angle A_i^{(p)}$ is the sum of the interior angles of p. Q.E.D.

2. If an n-gon is *regular*, then its n angles are all congruent. Let $\{\angle A_i\}_{i=1}^{n}$ be the angles of a regular n-gon p. Since p is regular, by definition $\angle A_1 \cong \angle A_2 \cong ... \cong \angle A_n$ and $m\angle A_1 = m\angle A_2 = ... = m\angle A_n$ since congruent angles have equal measure. Say $m\angle A_i = \alpha$.

   By the result in (1),

$$\sum_{i=1}^{n} \underbrace{m\angle A_i}_{\alpha} = 180(n-2)° = n\alpha,$$

   so we have for the measure of any of the angles:

$$\alpha = \frac{n-2}{n} 180°.$$

   Q.E.D.